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SOME PROBLEMS IN DESCRIBING VARIOUS PHYSICAL PROCESSES WITH SIMILAR NONLINEAR WAVE PROPAGATION MODELS

Abstract. The article provides a generalized presentation of the results of the authors' research to identify typical situations in the construction of mathematical models of various physical processes that can lead to equations with solutions in the form of moving nonlinear waves.

The problems of taking into account relaxation times and long-range interactions of structural elements of media in the mathematical description of the phenomena of mass, heat and momentum transfer are of great scientific and practical interest. The analysis shows that these issues are especially relevant when creating adequate mathematical models of high-intensity fast technological processes in conditions when the correctness of using methods of equilibrium thermodynamics becomes problematic. For the first time, sufficient conditions were established for the derivation of the perturbed KdV and Witham equations describing nonlinear wave propagation for transport phenomena in physico-chemical systems in the case of a low-intensity bottom mass source and with a quadratic relaxation function.

It is shown that the presence of a weak spatial nonlocality of the medium plays a fundamental role in the derivation of Whitham-type equations. The properties of a nonlocal integral relation for the flow of matter in a physicochemical system with a small deviation from the equilibrium state for a nonlinear relaxation have been established.

The kinetic equations of aggregation processes in the nonlocal form have been studied too. Comparing the obtained equations with the previously known kinetic equations for the aggregation processes, it can be concluded that account of different time delays for clusters of different orders significantly changes the form of the kinetic equations. This circumstance can especially manifest itself at the initial moment.

Key words: Physical and chemical processes, dissipative fluxes, transport equations, viscous liquids.

Introduction. The paper gives a generalized presentation of the authors' research results on identifying the typical situations while building mathematical models of various physical processes which can lead to equations with solutions in the form of moving nonlinear waves [1-4].

Examples of four different physical problems are considered, demonstrating a deep hidden analogy in the process of mathematical modeling. The derivations of the constitutive equations of the models and predictive analysis of the behavior of the solutions have been submitted. The methods for their asymptotic study developed by the authors applying to the considered problems have been also described.

The first problem is the derivation of a new basic equation for the propagation of nonlinear waves in an isothermal thin layer of an in viscid fluid in the presence of a source of low intensity mass on the reference surface of the flow and the active free surface, as well as the asymptotic study of solutions of the derived equation for describing the evolution of single nonlinear waves [4-6]. The novelty of the problem statement is due to the account of the activity of the free surface, which, in turn, is induced by the course of mass transfer processes at the interface between the liquid and vapor phases. This interpretation of surface activity opens up prospects for the practical use of the results of this section as the basis for engineering methods for calculating mass transfer equipment.

The second problem is to derive a new basic equation for wave propagation in viscous nonisothermal condensate films and to give an asymptotic study of its wave solutions [5-7].

The novelty of this formulation of the problem is due to taking into account the combined effect of the source of mass and appropriate increase in the flow rate of the liquid in the layer induced by the phase transition at the "liquid-vapor" interface, and of influence of the variable viscosity due to the non-isothermality of the liquid film.

The practical significance of this problem lies in the possibility of using the developed mathematical model as a basis for engineering methods for calculating condensation and evaporation processes in thin layers of viscous liquids [7, 8].

The third problem is the derivation and asymptotic analysis of the behavior of solutions of the equation described propagation of nonlinear waves in systems with nonlocal effects in the form of a modified integro-differential Whitham equation [9-12]. The novelty of the problem statement is due to the nonlinearity of the supposed relaxation function in the kernel of the integral operator [13, 14].

The fourth problem is the derivation and asymptotic study of nonlinear equations of evolution of the concentration of clusters of different orders in systems with nonlocality effects in the aggregation processes [15-20]. The novelty of the problem statement is due to taking into account synchronized and asynchronous delays in the growth of clusters, as well as taking into account the influence of their ages on aggregation activity.

The deep unity of the all four problems and appropriate mathematical models is due to the consideration of the influence of variable control parameters of mathematical models on the form of basic evolutionary equations describing the propagation of nonlinear waves in systems, as well as on the form and behavior of the solutions. The content of this unity is revealed when interpreting the variability of the control parameters in terms of the presence of sources and nonlocality effects.

Materials and methods. A review of the literature and an analysis of the known results show that both the increase in mass in the liquid flow and the non-isothermal nature of the process can have a great influence on the stability of the waveless regime of the thin layer liquid flow. This is especially true for flows accompanied by heat and mass transfer processes, as well as phase transitions [25-30].

The problem of the potential flow of a horizontal thin layer of an ideal fluid along the supporting surface with a weak source of mass at the bottom is considered. The equations describing such a flow with a free surface read as follows.

Continuity equation

$$\varphi_x + \varphi_y = 0. \quad (1)$$

Boundary condition on a solid wall (with no adhesion condition in the presence of a bottom mass source)

$$\varphi_x h_x + \varphi_y = q; \quad y = -h(x) \quad (2)$$

Dynamic boundary condition on a free surface

$$\varphi_t + q\eta + \frac{1}{2}\left(\varphi_x^2 + \varphi_y^2\right) = 0. \quad (3)$$

Kinematic boundary condition on a free surface

$$\eta_t + \varphi_x \eta_x - \varphi_y = 0, \quad (4)$$

where q is the potential of the fluid velocity; q - density of mass flux through a solid surface; $y = -h(x)$ - the equation describing the bottom profile; $\eta(x,t)$ - perturbed free surface profile.

The value of the mass flux density depends on the physical mechanism of the processes occurring in the bottom area. The simplest form of the corresponding functional dependence can be obtained from the condition:

$$V_n = kV_\tau. \quad (5)$$

Physically, a similar condition can be interpreted as the proportionality of the washout rate (i.e., the normal component of the fluid velocity immediately near the bottom) and the tangential component of the fluid velocity in the bottom area. This condition has a physical meaning, since the no-slip condition is not imposed for an ideal fluid.

Let's consider further long-wave approximation and introduce two small parameters:

$$h_0^2 / l^2 = \mu < 1.$$

$$a / h_0 = \varepsilon < 1.$$

Let us assume also that both small parameters are of the same order of smallness, and the intensity factor of the mass flow at the bottom is of a higher order, i.e.

$$k = k_1 \varepsilon \mu, \varepsilon = \mu . \tag{6}$$

Under the assumption of weak nonlinearity, the bottom profile should change slowly, i.e. $h = h(\varepsilon x)$.

The appropriate dimensionless variables read

$$x \to xl \ , \ \varphi \to \varphi \frac{a}{h_0}\sqrt{gh_0} \ , \ t \to t\frac{l}{\sqrt{gh_0}} \ , y \to yh_0, \ \eta \to \eta a , \ h \to hh_0, \tag{7}$$

Let's introduce now a special self-similar variable θ depending on slow coordinates $X = \varepsilon x , T = \varepsilon t$.

By eliminating the secular terms in the second and higher orders, we obtain the basic equation of wave propagation in the layer

$$U_X - \frac{3}{2}\frac{\theta_X \theta_T}{H} U_\theta + \frac{H\theta_X}{4}\left(\theta_T^2 - \frac{1}{3}H\theta_X^2\right)U_{\theta\theta\theta} = \left(\frac{\theta_T - H\theta_X + \theta_X (k_1 - H_X)}{2H\theta_X}\right)U , \tag{8}$$

For the obtained relation (8) to be satisfied in the zero order, the following equality, playing the role of the dispersion relation, should be fulfilled

$$\theta_T^2 - H\theta_X^2 = 0 . \tag{9}$$

Thus, for the accepted order of smallness of the density of the mass source at the bottom, its effect on the nature of the propagation of nonlinear waves on the surface of the layer is described within the framework of the general structure of the perturbed Korteweg-de Vries equation (when choosing $\theta_T < 0$ and $\theta_X > 0$).

If we assume a lower order of smallness for the source of mass, the structure of the evolutionary equations (8, 9) will be destroyed. In the following orders, we obtain a system of linear recurrent equations; these equations describe decreasing or increasing disturbances. In any case, the structure of the nonlinear evolutionary equations is destroyed, which can be interpreted as the damping effect of the mass source on the nonlinear wave regime.

The structure of the nonlinear evolutionary equation can change significantly under the influence of effects occurring on the free surface. Then, after similar transformations, from the conditions for excluding secular growth, we arrive at an evolutionary equation that differs from the perturbed Korteweg-de Vries equation obtained above.

$$U_X - \frac{3}{2}\frac{\theta_X \theta_T}{H} U_\theta + \frac{H\theta_X}{4}\left(\theta_T^2 - \frac{1}{3}H\theta_X^2\right)U_{\theta\theta\theta} =$$

$$= \left(\frac{\theta_T - H\theta_{XX} + \theta_X (k_1 - H_X)}{2H\theta_X}\right)U + \frac{\sigma\theta_T^3}{2H\theta_X}U_{\theta\theta} . \tag{10}$$

Using the expansion of differential operators in the vicinity of the critical values of the control parameters and then excluding the secular terms, we arrive at amplitude equations (3) of the type that describe various nonlinear wave processes with dispersion. However, there are no soliton solutions to such equations.

The stability of such wave flows depends on the order of the control parameter

$$K_{sf} = \frac{\sigma\theta_T^3}{2H\theta_X} . \tag{11}$$

As a conclusion from this analysis and (11), we can conclude that the nature of propagation and evolution of nonlinear waves in systems with sources of mass depends significantly both on the order of the intensity of the sources and on the type of boundary conditions. Moreover, in different situations, even the form of evolutionary equations can change significantly [6,8,10].

Evolution equations for nonlinear waves in viscous condensate films

Earlier it was shown that when a condensate film flows down, a situation may arise when the stationary Nusselt problem has no solution, and it can be assumed that nonlinear waves can be generated in regions of high temperature and viscosity gradients. The complexity of the analysis of wave solutions in the case of film condensation is that in the presence of a mass source, the film consumption increases. As a result, there are no constant solutions. It is also necessary to take into account the fact that in the mathematical modeling of film condensation we are dealing with an essentially dissipative system and, secondly, the resulting systems of equations cannot be decoupled in principle due to the presence of sources.

The equations of motion and continuity in the long-wave approximation:

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y}\right)+g_{ef}+\frac{\sigma}{\rho}\frac{dK_S}{dx}. \quad (12)$$

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0. \quad (13)$$

Here K_s is the surface curvature.

Equation of material balance for condensate

$$\frac{\partial h}{\partial t}+\frac{\partial j}{\partial x}=I_m, \quad (14)$$

The intensity of the source of mass reads

$$I_{m,(con)}=\frac{\lambda}{r\rho}\frac{\partial T}{\partial y}\bigg|_{y=h}$$

Integrating the equation of motion over the thickness, after a series of transformations, we obtain

$$\frac{\partial j}{\partial t}+\frac{\partial}{\partial x}\int_0^h U^2dy-U_S\left(\frac{\partial h}{\partial t}+\frac{\partial j}{\partial x}\right)=-\nu_w\frac{\partial U}{\partial y}\bigg|_{y=0}+g_{ef}\,h+\frac{\sigma h}{\rho}\frac{dK_S}{dx}. \quad (15)$$

Let’s looking for the velocity profile over the film thickness in the form

$$U=U_S f(\eta,x;h).$$

The resulting integral evolutionary relation linked the film thickness and flow rate reads

$$\frac{\partial j}{\partial t}+\frac{\partial}{\partial x}\left(\frac{f_2}{f_1^2}\frac{j^2}{h}\right)+\frac{j}{f_1h^2}\left(f_3\nu_w-I_mh\right)=g_{ef}h+\frac{\sigma h}{\rho}\frac{dK_S}{dx}. \quad (16)$$

At a sufficient distance from the initial point, the intensity of the mass source during film condensation is, as a rule, low. This made it possible to introduce into consideration a new small parameter $\varepsilon=\lambda\Delta T/r\rho\langle j_0\rangle$ in the work, where $\langle j_0\rangle$ is the averaged flow rate of condensate in the undisturbed film in the area under consideration. This approach is additionally justified due to the large values of the phase transition heat.

To construct mathematical models capable of describing the evolution of wave perturbations of the condensate film profile, an asymptotic analysis of (5) was carried out using the methods of secular perturbation theory.

In contrast to previously known works, second-order terms are retained to describe the evolution of a wave packet in the weakly nonlinear approximation. As a result, we get

$$\frac{\partial j_1}{\partial t}+\alpha_1\frac{\partial j_1}{\partial x}+\alpha_2\frac{\partial h_1}{\partial x}+\alpha_3\frac{\partial^3 h_1}{\partial x^3}+\alpha_4 j_1+\alpha_5 h_1=\beta_1 j_1\frac{\partial j_1}{\partial x}+\beta_2 h_1\frac{\partial h_1}{\partial x}+\beta_3 j_1^2+\beta_4 h_1^2,$$

$$X = \varepsilon x, \quad T = \varepsilon t \cdot \tag{17}$$

If the temperature of the support surface is not constant, then it is necessary to include the dependence on the film thickness in the expression for the self-similar velocity profile [12].

The paper establishes the following features of the developed mathematical model.

First, in addition to the usual convective nonlinearities, nonlinear terms appear in the equations due to the pumping of energy and mass due to an increase in the flow rate of the liquid in the film.

Secondly, energy is pumped into the system due to another source - gravitational forces. Due to this and other reasons, the resulting system cannot be decoupled within the framework of formal mathematical calculations.

However, in this work, this was done on the basis of the results of the analysis of the linearized problem and remaining within the framework of an adequate description of the qualitative behavior of small perturbations of the stationary solution.

There were introduced new stretched variables $X = \varepsilon x, \quad T = \varepsilon t$ and fast variable $\eta = \theta(X,T)/\varepsilon$.

In this work, the conditions for the solvability of the resulting system are established in the form of a new dispersion relation:

$$\begin{vmatrix} \frac{\partial\theta}{\partial T} + \alpha_1 \frac{\partial\theta}{\partial X} + \alpha_4 & \alpha_2 \frac{\partial\theta}{\partial X} + \alpha_3 \left(\frac{\partial\theta}{\partial X}\right)^3 + \alpha_5 \\ \frac{\partial\theta}{\partial X} & \frac{\partial\theta}{\partial T} - z_1 \end{vmatrix} = 0. \tag{18}$$

As a result, the new basic equation for the evolution and spatial variation of the condensate film thickness has been obtained

$$\frac{\partial h_1}{\partial t} + \frac{\beta_1 L + \beta_2/L}{\alpha_1 + \alpha_2/L} h_1 \frac{\partial h_1}{\partial \xi} - \frac{\alpha_3}{\alpha_1 + \alpha_2/L} \frac{\partial^3 h_1}{\partial \xi^3} = -(\alpha_4 + \alpha_5/L)h_1 + (\beta_3 L + \beta_4/L)h_1^2 \cdot \tag{19}$$

The newly obtained equations (18,19) is structurally close to the Korteweg-de-Vries equation with a nonlinear perturbation of the right-hand side and slowly varying coefficients. The presence of such a perturbation leads to the fact that the dispersion relation of the last equation contains a nonzero imaginary part, and an undamped wave solution can exist only on the neutral line and in the region of increasing amplitudes. According to the well-known classification, the instability of the solution to the obtained problem belongs to the category of dissipative instability.

Results. The problems of taking into account relaxation times and long-range interactions of structural elements of media in the mathematical description of the phenomena of transfer of mass, heat and momentum are of great scientific and practical interest. Similar problems arise when describing the development of internal stresses and the formation of cracks in solids. The analysis shows that these issues are especially relevant when creating adequate mathematical models of high-intensity fast technological processes in conditions when the correctness of using the methods of equilibrium thermodynamics becomes problematic. It was previously discovered [12] that an equation of this type can be obtained by describing the propagation of nonlinear waves in media with spatial nonlocality by the method of relaxation transfer kernels in the case of a linear relaxation function. However, this assumption does not agree well with the general nonlinear nature of the developed models.

The main scientific contribution of this section is that sufficient assumptions were established for the correct derivation of the modified Whitham equation describing the nonlinear propagation of waves in transport phenomena in physicochemical systems. It is shown that the presence of spatial nonlocality of the medium can play a fundamental role in the derivation of Whitham-type equations [2, 3, 4]. The questions of physical interpretation of the investigated model are also considered. It is shown that the manifestation of nonlocal effects in the systems under study can be substantiated and obtained as a result of the presence of domains with a complex spatial structure, as well as the presence of heat and mass sources.

The driving force of substance transfer is the local deviation of the control parameter u of the process, which characterizes the state of equilibrium of the system. Temperature is such a parameter for thermal processes; for mass transfer processes - chemical potential; for the propagation of internal defects in solids, these are equilibrium internal stresses.

$\Delta \nu = u$.

Then the expression for the flow of matter at small deviations from equilibrium, but taking into account various kinds of non-local effects, in particular, in media with memory, can be written in the form [12]:

$$J = \int_{\Omega} N(\theta,u) \nabla u(s,t) ds \ .$$

Here N is the kernel of the integral operator,

We denote the derivative of the kernel in the integral operator of the equation as

$$G(\theta,u) = \frac{\partial N(\theta,u)}{\partial \theta} \ .$$

The use of the conservation law leads to the following equation

$$u_t = \nabla \int_{\Omega} \left(\sum_k G_{(k)} u^{k+1} \right) ds = I \ . \qquad (20)$$

For the successful implementation of further transformations, it is necessary to specify the commutation condition for the differentiation and convolution operators in equation (1). Since at this stage of transformations the form of the kernels of the integral operator is unknown, this condition will need to be additionally checked for a specific type of physically significant kernels. Then equation (20) can be rewritten as

$$u_t + \int_{\Omega} \left(\sum_k (k+1) G_{(k)} u^k \right) u_s ds = I \ . \qquad (21)$$

Further development of the theory requires a refinement of the form of the kernels of the operator in equation (21). In order not to violate the logic of the nonlinear approach, in our work for the first time, in contrast to the work of Brener [12], the relaxation equation is written in a general form.

$$\frac{d}{d\theta} G_{(k)}(\theta) + B_{(k)} \Phi\left(G_{(k)}(\theta)\right) = 0 \ . \qquad (22)$$

$\Phi(\bullet)$ must be a positive non-decreasing function.

Then it can be shown that the physically significant form of the equation looks as follows

$$\frac{d}{d\theta} G_{(k)}(\theta) + Y_{1,(k)} G_{(k)}(\theta) \pm Y_{2,(k)} G_{(k)}^2(\theta) = 0 \ , \qquad (23)$$

where, $Y_{1,(k)} \geq 0$ and $Y_{2,(k)} \geq 0$.

It can be shown that since the kernels of the integral operator in the transport equation are obtained under the assumption of weak nonlocality, the system (21, 23) has a natural small parameter $\varepsilon = r_{(0)}/R$.

Here $r_{(0)}$ is the maximum radius that should be taken into account when describing the nonlocal interaction, R is the characteristic scale of the macroscopic system.

It was proved by us with the help of the detail consideration that the commutation condition for the differentiation and convolution operators for the kernels of all the considered quadratic forms is fulfilled.

In accordance with the chosen strategy of excluding members of a higher than the second order, the following basic equation was obtained

$$u_t + \int_{\Omega} G_{(0)}(\theta) u_s ds + 2 \int_{\Omega} G_{(1)}(\theta) u u_s ds = I \ . \qquad (24)$$

The fundamental moment and scientific contribution of the performed modeling of relaxation kernels is that a wider class of relaxation functions was considered than was done earlier.

Taking into account the specifics of the behavior of all the considered types of kernels in integral operators under the restriction of a sufficiently fast decrease in $r_{(k)}$ with an increase in the number k, the following transfer equation has been derived

$$u_t + 2\chi G^0_{(1)} u u_x + \int\limits_{\Omega} G_{(0)}(x-s) u_s ds = I. \qquad (25)$$

where χ is the normalizing coefficient.

Having defined the spatial variable $\varsigma = \dfrac{x}{2\chi G^0_{(1)}}$, the previous equation can be rewritten in the form of perturbed Whitham's equation

$$u_t + u u_\varsigma + \int\limits_{\Omega} G_{(0)}(\varsigma - s) u_s ds = I. \qquad (26)$$

With such a rearrangement, the appearance of the kernels does not fundamentally change. At $I = 0$ equation (34) takes the form (20) of the usual Whitham equation.

Further, after a number of cumbersome but simple in mathematical technique rearrangements, the following ordinary differential equation succeed to be obtained

$$(c - u_0)^2 \frac{d^2 u_0}{d\xi^2} = \frac{\varphi_{(0)} u_0^2}{r_{(0)}} \left[\frac{\varphi_{(0)} u_0^2}{\alpha r_{(0)}} + \left(\beta G^0_{(0)} - \frac{c\varphi_{(0)}}{2r_{(0)}} \right) u_0 + c \left(\frac{c\varphi_{(0)}}{2r_{(0)}} - G^0_{(0)} \right) \right], \qquad (27)$$

where the parameters $\alpha,\ \beta$ are depend on the type of relaxation function.

The subsequent analysis using the phase plane method showed that equations of this type (27) have solutions in the form of a solitary traveling wave capable of propagating over considerable distances with a slight change in profile [2, 4].

4. Nonlocal mathematical models of aggregation processes in dispersed media

Particle aggregation is widespread in various chemical engineering processes, metallurgy and nature, and there are many approaches to modeling this phenomenon. At the same time, some important aspects of the description of aggregation processes have not yet been developed. One of such important, but poorly worked out issues is the nonlocality of aggregation processes in time.

However, without taking this aspect into account, it is impossible to describe the effect of the characteristic relaxation times of the formation of aggregates on the kinetics of the process. This is especially true when applied to nanotechnological processes.

The problem considered in the section is devoted to the development and analysis of a nonlocal modification of the Smoluchowski equation, which is the basic model of the kinetics of aggregation processes in dispersed systems. The work is based on a nonlocal model based on the Smoluchowski equation, proposed earlier in the works [17, 18].

At the same time, it is firstly in this work, the cases of synchronous and asynchronous delays in the formation of aggregates - clusters of different orders in a single system are considered separately.

The generalized model in the form of an integro-differential equation is as follows:

$$\frac{dC_i}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} \int_0^t \int_0^t N_{j,i-j} C_j(t_1) C_{i-j}(t_2) dt_1 dt_2 - \sum_{j=1}^{\infty} \int_0^t \int_0^t N_{i,j} C_i(t_1) C_j(t_2) dt_1 dt_2 \qquad (28)$$

C_i denote the concentrations of i -mers, and aggregation kernels $N_{i,j}$ are functions of the delay times $(t - t_1)$ and $(t - t_2)$.

In our case, the characteristic times of aggregation and measures play the role of relaxation times. The simplest model equation for the elements of the aggregation matrix can be constructed by analogy with the model equation for the transfer kernels:

$$r_i \frac{\partial N_{i,j}}{\partial s_i} + r_j \frac{\partial N_{i,j}}{\partial s_j} + \frac{f^0_{i,j}}{\tau_{i,j}} N_{i,j} = 0, \qquad (29)$$

where $s_j = t - t_2$ $s_j = t - t_2$

In equation (29), the coefficients τ_{ij}, along with the relaxation times τ_{ij}, play the role of control parameters of the "inertness" of clusters; the parameter f is responsible for the characteristics of media and particles.

The aggregation matrix satisfying Eq. (28), in the case of fast relaxation, has

$$N_{i,j} = \eta^0_{i,j} \exp\left(-\frac{f^0_{i,j}}{2\tau_{i,j}} \left(\frac{s_i}{r_i} + \frac{s_j}{r_j} \right) \right)$$

On the basis of the approach of relaxation transfer kernels, the main governing equation was obtained in this work:

$$\frac{dC_i}{dt} = \frac{1}{2}\sum_1 \eta_{j,i-j} \exp(-(g^{(j)}_{j,i-j} + g^{(i-j)}_{j,i-j})t) I_1 I_2 - \sum_2 \eta_{i,j} \exp(-(g^{(i)}_{i,j} + g^{(j)}_{i,j})t) I_3 I_4, \quad (30)$$

Here $I_1 = \int_0^t \exp(g^{(i-j)}_{j,i-j}s) C_{i-j}(s)ds$; $I_2 = \int_0^t \exp(g^{(j)}_{j,i-j}s) C_j(s)ds$; $I_3 = \int_0^t \exp(g^{(j)}_{i,j}s) C_j(s)ds$

$I_4 = \int_0^t \exp(g^{(i)}_{i,j}s) C_i(s)ds$.

Discussion. Asynchrony in the formation of clusters of different orders means that there is no correlation between the relaxation times t_1, t_2 in equation (28).

The subsequent analysis was based on the asymptotes of the integrals in (30). Namely, it is assumed that for small relaxation times, the Laplace method can be used in the vicinity of the time instant. But the immediate substitution of the expansion of the integrals in equation (30) requires the multiplication of asymptotic sequences. Such a procedure is dangerous, as it can lead to a complete loss of order when checking the approximation.

To solve this problem, a specific method has been developed by us that made it possible to reduce equation (28) to a form free of the product of integrals:

$$\frac{d^2C_i}{dt^2} + a\frac{dC_i}{dt} = \frac{1}{2}\exp\left(-\frac{at}{2}\right)\sum_1 \eta_{j,i-j}(C_j I_1 + C_{i-j} I_2) - \exp\left(-\frac{at}{2}\right)\left[C_i \sum_2 \eta_{i,j} I_2 + I_3 \sum_2 \eta_{i,j} C_j \right]. \quad (31)$$

Further, using the Laplace method, we obtain asymptotic relations in which the orders of equations and approximations are concerted.

As a result of further transformations, accompanied by a comparison of the smallness of different relaxation times, the reduced master kinetic equation was obtained for the first time in this work

$$\varepsilon\frac{d^2C_i}{d\theta^2} + \frac{dC_i}{d\theta} = 2\varepsilon^2 \sum_1 \overline{\eta}_{j,i-j}\left[C_j C_{i-j} - \varepsilon\frac{d}{d\theta}(C_j C_{i-j}) \right] - 4\varepsilon^2 \sum_2 \overline{\eta}_{i,j}\left[C_i C_j - \varepsilon \frac{d}{d\theta}(C_i C_j) \right]. \quad (32)$$

Here $\varepsilon = \tau_* / T$, $\frac{1}{a} = \tau_*$, T is the characteristic time of the process, $\theta = t/T$ is the dimensionless time $\theta = t/T$, and dimensionless aggregation kernels $\overline{\eta}_{i,j} = T^3 \eta_{i,j}$.

The characteristic time of the initial period of the aggregation process, during which the greatest influence of the nonlocality of the process is observed, reads:

$\Delta\theta_{in} \sim -\varepsilon \ln \varepsilon$.

Modified model for the case of synchronous delays

As a special case, the paper considers a modification of the Smoluchowski equation with a synchronous time delay of aggregation of clusters of different orders, which is intended to describe the effect of the characteristic time of the formation of aggregates on the kinetics of the process.

In this case, the master integro-differential equation takes the form [17]:

$$\frac{\partial C_i}{\partial t} = \frac{1}{2}\sum_{j=1}^{i-1} \int dt_1 \Phi^0_{i-j,j} \exp\left(-\frac{f^0_{i-j,j}}{\tau_{i-j,j}}(t - t_1) \right) C_{i-j}(t_1) C_j(t_1) - \sum_{j=1}^{\infty} \int dt_1 \Phi^0_{i,j} \exp\left(-\frac{f^0_{i,j}}{\tau_{i,j}}(t - t_1) \right) C_i(t_1) C_j(t_1) \quad (34)$$

Using the procedure for separate averaging over separate groups of indices the specific kinetic equation of the third order in time has been obtained.

The compact form of this equation reads

$$\frac{d^3C_i}{dt^3}+\left(B_1+B_2\right)\frac{d^2C_i}{dt^2}+B_1B_2\frac{dC_i}{dt}=(B_1+B_2+\frac{d}{dt})(\frac{1}{2}\sum_{j=1}^{i-1}\Phi^0_{i-j,j}C_{i-j}(t)C_j(t)-\sum_{j=1}^{\infty}\Phi^0_{i,j}C_i(t)C_j(t))-$$
$$-\frac{1}{2}A_1\sum_{j=1}^{i-1}\Phi^0_{i-j,j}C_{i-j}(t)C_j(t)+A_2\sum_{j=1}^{\infty}\Phi^0_{i,j}C_i(t)C_j(t) \qquad (35)$$

A feature of the obtained equation (35) is the presence of solutions describing the propagation of perturbations with a finite velocity in the form of solitary waves [8].

Conclusion. In the submitted work, for the first time sufficient conditions were established for the derivation of the perturbed KdV and Whitham equations describing the nonlinear propagation of waves for transport phenomena in physicochemical systems in the case of bottom mass source of low intensity and under the quadratic relaxation function.

It is shown that the presence of a weak spatial nonlocality of the medium plays a fundamental role in the derivation of Whitham-type equations. The properties of a nonlocal integral relation for the flow of matter in a physicochemical system with a small deviation from the equilibrium state for a nonlinear relaxation have been established.

The kinetic equations of aggregation processes in the nonlocal form have been studied too. Comparing the obtained equations with the previously known kinetic equations for the aggregation processes, it can be concluded that account of different time delays for clusters of different orders significantly changes the form of the kinetic equations. This circumstance can especially manifest itself at the initial moment.

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ТОЛҚЫНДАРДЫҢ ТАРАЛУЫНЫҢ ҰҚСАС СЫЗЫҚТЫ ЕМЕС МОДЕЛЬДЕРІН ҚОЛДАНА ОТЫРЫП, ӘРТҮРЛІ ФИЗИКАЛЫҚ ПРОЦЕСТЕРДІ СИПАТТАУДЫҢ КЕЙБІР МӘСЕЛЕЛЕРІ

Аннотация. Мақалада қозғалмалы сызықты емес толқындар түріндегі шешімдері бар теңдеулерге әкелуі мүмкін әртүрлі физикалық процестердің математикалық модельдерін құру кезіндегі типтік жағдайларды анықтау бойынша авторлардың зерттеу нәтижелерінің жалпыланған шешімдері келтірілген. Массаның, жылу мен импульстің берілу құбылыстарын математикалық сипаттау кезінде релаксация уақытын және қоршаған ортаның құрылымдық элементтерінің ұзақ мерзімді өзара әрекеттесуін ескеру мәселелері үлкен ғылыми және практикалық қызығушылық тудырады.

Талдау көрсеткендей, бұл сұрақтар тепе-теңдік термодинамика әдістерін қолданудың дұрыстығы проблемалы болған жағдайда жоғары қарқынды жылдам технологиялық процестердің жеткілікті математикалық модельдерін құруда өте маңызды. Алғаш рет төмен қарқындылықтағы төменгі масса көзі жағдайында және квадраттық релаксация функциясы бар физика-химиялық жүйелердегі тасымалдау құбылыстары үшін толқындардың сызықтық емес таралуын сипаттайтын КдВ және Уизем теңдеулерін алу үшін жеткілікті шарттар жасалды.

Уитхем типті теңдеулерді шығаруда әлсіз кеңістіктік жергілікті емес ортаның болуы негізгі рөл атқаратындығы көрсетілген. Сызықтық емес релаксация үшін тепе-теңдік күйінен аз ауытқумен физика-химиялық жүйеде зат ағымы үшін локальды емес интегралдық қатынастардың қасиеттері анықталды. Жергілікті емес формадағы агрегация процестерінің кинетикалық теңдеулері де зерттелді. Алынған теңдеулерді агрегация процестері үшін бұрын белгілі болған кинетикалық теңдеулермен салыстыра отырып, әртүрлі ретті кластерлер үшін әртүрлі уақыттық кідірістерді есепке алу кинетикалық теңдеулердің пішінін айтарлықтай өзгертеді деп қорытынды жасауға болады. Бұл жағдай әсіресе бастапқы сәтте пайда болуы мүмкін.

Түйінді сөздер: физика-химиялық процестер, диссипативті ағындар, тасымалдау теңдеулері, тұтқыр сұйықтықтар.

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НЕКОТОРЫЕ ПРОБЛЕМЫ ОПИСАНИЯ РАЗЛИЧНЫХ ФИЗИЧЕСКИХ ПРОЦЕССОВ С ПОМОЩЬЮ АНАЛОГИЧНЫХ НЕЛИНЕЙНЫХ МОДЕЛЕЙ РАСПРОСТРАНЕНИЯ ВОЛН

Аннотация. В статье представлено обобщенное представление результатов исследований авторов по выявлению типичных ситуаций при построении математических моделей различных физических процессов, которые могут приводить к уравнениям с решениями в виде движущихся нелинейных волн. Проблемы учета времен релаксации и дальнодействующих взаимодействий структурных элементов сред при математическом описании явлений передачи массы, тепла и импульса представляют большой научный и практический интерес.

Анализ показывает, что эти вопросы особенно актуальны при создании адекватных математических моделей высокоинтенсивных быстрых технологических процессов в условиях, когда корректность использования методов равновесной термодинамики становится проблематичной. Впервые были созданы достаточные условия для вывода возмущенных уравнений КдВ и Уизема описывающих нелинейное распространение волн для явлений переноса в физико-химических системах в случае источника донной массы низкой интенсивности и с квадратичной функцией релаксации.

Показано, что наличие слабой пространственной нелокальности среды играет фундаментальную роль при выводе уравнений типа Уитхема. Установлены свойства нелокального интегрального соотношения для потока вещества в физико-химической системе с небольшим отклонением от равновесного состояния для нелинейной релаксации. Также были изучены кинетические уравнения процессов агрегации в нелокальной форме. Сравнивая полученные уравнения с ранее известными кинетическими уравнениями для процессов агрегации, можно сделать вывод, что учет различных временных задержек для кластеров разного порядка существенно изменяет форму кинетических уравнений. Это обстоятельство может особенно проявиться в начальный момент.

Ключевые слова: физико-химические процессы, диссипативные потоки, уравнения переноса, вязкие жидкости.

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REFERENCES

[1] Karpman V.I. 2011 *Nonlinear Waves in Dispersive Media* (Oxford, New York, Toronto, Sydney: Pergamon Press).

[2] Brener A.M., Tashimov L. 2001 Wave Regimes of the Vapour Film Condensation *Computational Modelling of Free and Moving Boundary Problems* (Southampton) pp 51-60.

[3] Tseluiko D., Kalliadasis S. 2011 *J. Fluid Mech.* vol 673 (Cambridge University Press) p 19.

[4] Jou D., Casas-Vazquez J., and Criado-Sancho M., Thermodynamics of Fluids under Flow, Berlin: Springer, 2001, p. 231.

[5] Brener A.M. Nonlocal Equations of the Heat and Mass Transfer in Technological Processes Theoretical Foundations of Chemical Engineering, 2006, Vol. 40, No. 6, pp. 564–572.

[6] J.A.D. Wattis. An introduction to mathematical models of coagulation-fragmentation processe: A discrete deterministic mean-field approach. Physica D 222 (2006), 1-20.

[7] Brener A., Makhatova A., Yakubova R. Modeling of aggregation processes in multiphase chemical reactors. Proc.of 11th Int. Conf. on Multiphase Flow in Ind. Plants, Palermo, 2008, 611-618.

[8] Trifonov Yu. 2017 *Physics of Fluid* vol 29 (American Institute of Physics) p 1.
[9] Abcha N., Pelonovsky E., Mutabazi I. 2018 *Nonlinear Waves and Pattern Dynamics* (Cham: Springer).
[11] Tsvelodub O.Yu., Arkhipov D.G. 2017 *Journal of Applied Mechanics and Technical Physics* vol 58 No 4 p 619.
[12] Archilla J.F.R., Zolotaryuk Ya., Kosevich Ya., Doi Yu. 2018 *CHAOS* vol 28 (AIP Publishing) 083119.
[13] Yazhou Shi., Xiangpeng Li., Ben-gong Zhang 2018 *Advances in Mathematical Physics* vol 2018. p 31.

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