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Әл-Фараби атындағы Қазақ ұлттық университетінің

# Х А Б А Р Л А Р Ы

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## ИЗВЕСТИЯ

НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК  
РЕСПУБЛИКИ КАЗАХСТАН  
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## NEWS

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NAS RK is pleased to announce that News of NAS RK. Series physico-mathematical journal has been accepted for indexing in the Emerging Sources Citation Index, a new edition of Web of Science. Content in this index is under consideration by Clarivate Analytics to be accepted in the Science Citation Index Expanded, the Social Sciences Citation Index, and the Arts & Humanities Citation Index. The quality and depth of content Web of Science offers to researchers, authors, publishers, and institutions sets it apart from other research databases. The inclusion of News of NAS RK. Series of chemistry and technologies in the Emerging Sources Citation Index demonstrates our dedication to providing the most relevant and influential content of chemical sciences to our community.

Қазақстан Республикасы Ұлттық ғылым академиясы «ҚР ҰҒА Хабарлары. Физикалық-математикалық сериясы» ғылыми журналының Web of Science-тің жаңаланған нұсқасы Emerging Sources Citation Index-те индекстелуге қабылданғанын хабарлайды. Бұл индекстелу барысында Clarivate Analytics компаниясы журналды одан әрі the Science Citation Index Expanded, the Social Sciences Citation Index және the Arts & Humanities Citation Index-ке қабылдау мәселесін қарастыруда. Web of Science зерттеушілер, авторлар, баспашылар мен мекемелерге контент тереңдігі мен сапасын ұсынады. ҚР ҰҒА Хабарлары. Химия және технология сериясы Emerging Sources Citation Index-ке енуі біздің қоғамдастық үшін ең өзекті және беделді химиялық ғылымдар бойынша контентке адалдығымызды білдіреді.

НАНПК сообщает, что научный журнал «Известия НАНПК. Серия физико-математическая» был принят для индексирования в Emerging Sources Citation Index, обновленной версии Web of Science. Содержание в этом индексировании находится в стадии рассмотрения компанией Clarivate Analytics для дальнейшего принятия журнала в the Science Citation Index Expanded, the Social Sciences Citation Index и the Arts & Humanities Citation Index. Web of Science предлагает качество и глубину контента для исследователей, авторов, издателей и учреждений. Включение Известия НАНПК в Emerging Sources Citation Index демонстрирует нашу приверженность к наиболее актуальному и влиятельному контенту по химическим наукам для нашего сообщества.

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E-mail: [ibraimova@aphi.kz](mailto:ibraimova@aphi.kz)**EVOLUTION EQUATIONS OF THE RESTRICTED THREE-BODY  
PROBLEM WITH VARIABLE MASSES**

**Abstract.** In classical celestial mechanics, the theory of perturbation based on Keplerian motion is well developed. Various kinds of perturbed motion equations are described by canonical equations, Lagrange equations and Newton's equations. The masses of bodies are assumed constant and the bodies are point or spherical. Classical perturbation theory is a basic tool in the study of many astronomical problems. Observational astronomy shows that the masses of real cosmic bodies change over time. Real celestial bodies are neither spherical nor solid. Celestial bodies are non-stationary and their masses, sizes, shapes and structures change over time as they evolve. There are now a number of studies with a community bibliography of papers on the celestial mechanics of bodies of variable mass. However, the perturbation theory for unsteady space systems is not sufficiently developed. Therefore, we considered a gravitational system consisting of three spherical celestial bodies, with spherical density distributions, with variable masses in the restricted formulation. We have considered the general case where the masses of the bodies change non-isotropically at different rates, in the presence of reactive forces. We will assume that a body with infinitesimal mass has no effect on the motions of the two main bodies. The problem was investigated by methods of perturbation theory based on aperiodic motion along the quasi-conic section developed for nonstationary gravitating systems. Evolution motion equations of a less massive body and a small body in oscillating elements, in the relative coordinate system, have been obtained.

**Key words:** restricted three-body problem, non-isotropic mass change, reactive forces, aperiodic motion along a quasi-conic section, perturbed motion equations in the form of Newton's equation.

**1. Introduction.** Modern celestial mechanics mainly applies the perturbation theory based on Keplerian motion. Classical perturbation theory is the basic tool in the study of many astronomical problems. Various kinds of perturbed motion equations are described by the canonical equations, the Lagrange and Newton equations. In this case, the masses of bodies are assumed constant, the bodies are point or spherical. However, observational astronomy shows that real celestial bodies are neither spherical nor solid. Their masses, sizes, shapes and structures change over time.

The novelty of our work is that we considered a model problem proposed as an initial approximation for the problems of celestial mechanics of bodies of variable mass in the form of Newton's equations. On the basis of this model problem, methods of perturbation theory are developed and new forms of perturbed motion equations of the restricted three-body problem with variable masses in the presence of reactive forces are obtained.

**2. Problem statement and motion equations in the relative coordinate system.** Consider a gravitational system consisting of three spherical celestial bodies, with spherical (or point) density distributions, with variable masses. Let,  $T_0$  be the more massive body,  $T_1$  the less massive body, then the main bodies. Let us denote by  $T_2$  the body with a very small mass, then the body with a small mass. Accordingly, we denote the masses, which are functions of time and are considered known

$$m_0 = m_0(t), \quad m_1 = m_1(t), \quad m_2 = m_2(t). \quad (2.1)$$

Let's assume that the masses of bodies decrease due to separating particles and increase due to joining particles. In this case, in general, the relative velocity of the separating particles from the body differs from the relative velocity of the joining particles to the body. Let us consider the general case when the masses of bodies change non-isotropically at different rates, in the presence of reactive forces [1]

$$\frac{\dot{m}_0}{m_0} \neq \frac{\dot{m}_1}{m_1}, \quad \frac{\dot{m}_0}{m_0} \neq \frac{\dot{m}_2}{m_2}, \quad \frac{\dot{m}_1}{m_1} \neq \frac{\dot{m}_2}{m_2}. \quad (2.2)$$

Let's assume that the body with a small mass  $m_2$ , does not affect to the motion of the two main bodies with masses  $m_0$  and  $m_1$ , and this is the statement restricted of the considered problem of three bodies with variable masses

$$m_2 \ll m_0, \quad m_2 \ll m_1. \quad (2.3)$$

The motion equations of the two main bodies in the relative coordinate system, with the origin in the center of the body  $T_0$  (the central body) will be written in the form

$$\ddot{\vec{r}}_1 + f(m_0 + m_1) \frac{\vec{r}_1}{r_1^3} = \vec{F}_1, \quad (2.4)$$

$$\vec{F}_1 = \vec{F}_1(t), \quad r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}. \quad (2.5)$$

Accordingly, the motion equations of a infinitesimal mass body in the Newtonian field of attraction of two main bodies in the relative system of coordinates looks like

$$\ddot{\vec{r}}_2 + fm_0 \frac{\vec{r}_2}{r_2^3} = \vec{\Pi}_2 + \vec{F}_2, \quad (2.6)$$

$$\vec{\Pi}_2 = \text{grad}_{\vec{r}_2} U, \quad U = fm_1 \left( \frac{1}{r_{12}} - \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{r_1^3} \right), \quad r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}, \quad (2.7)$$

$$\vec{F}_2 = \vec{F}_2(t), \quad r_{12} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = r_{21}. \quad (2.8)$$

In equations (2.4) and (2.6)  $f$  is the gravitational constant,  $\vec{F}_1, \vec{F}_2$  is the net reactive force.

In the orbital coordinate system the perturbing forces (2.5), (2.7), (2.8) can be written in the form

$$\vec{F}_1 = \vec{F}_1(t) = F_{1r}(t) \cdot \vec{e}_r + F_{1\tau}(t) \cdot \vec{e}_\tau + F_{1n}(t) \cdot \vec{e}_n, \quad (2.9)$$

$$\vec{F}_2 = \vec{F}_2(t) = F_{2r}(t) \cdot \vec{e}_r + F_{2\tau}(t) \cdot \vec{e}_\tau + F_{2n}(t) \cdot \vec{e}_n, \quad (2.10)$$

$$\vec{\Pi}_2 = \vec{\Pi}_2(t, x_1, y_1, z_1, x_2, y_2, z_2) = \Pi_{2r} \cdot \vec{e}_r + \Pi_{2\tau} \cdot \vec{e}_\tau + \Pi_{2n} \cdot \vec{e}_n, \quad (2.11)$$

where  $\vec{F}_{1r}$  the radial (directed along the radius vector),  $\vec{F}_{1\tau}$  transversal (perpendicular to the radius vector, lying on the plane of the instantaneous orbit and directed toward the motion side) and  $\vec{F}_{1n}$  normal (perpendicular to the plane of the instantaneous orbit) components of the reactive forces, similarly for  $\Pi_{2r}, \Pi_{2\tau}, \Pi_{2n}$ . The moving orthogonal trihedron of unit vectors  $\vec{e}_r, \vec{e}_\tau, \vec{e}_n = \vec{e}_r \times \vec{e}_\tau$  form the right triplet [2]-[6].

The obtained motion equations of the restricted three-body problem with variable masses in the presence of reactive forces (2.4), (2.6) are in general very difficult. Therefore we investigate the problem by the methods of the perturbation theory based on the aperiodic motion along the quasi-conic section developed for such nonstationary gravitating systems [7].

**3. Perturbation theory based on aperiodic motion along a quasi-conic section.** A number of problems in celestial mechanics of bodies with variable masses can be described by equations of the form as the two-body problem [8] with variable masses in the presence of perturbing forces

$$\ddot{\vec{r}} + f v \frac{\vec{r}}{r^3} - \tilde{a} \dot{\vec{r}} - \tilde{b} \vec{r} = \vec{W}, \quad (3.1)$$

where  $\vec{r} = \vec{r}(x, y, z)$ - radius vector,  $\dot{\vec{r}} = \vec{v}$  - velocity,  $v = \tilde{v}_1(t) + \tilde{v}_2(t) = v(t)$ ,  $\tilde{a} = \tilde{a}(t)$ ,  $\tilde{b} = \tilde{b}(t)$  given functions of their arguments.  $\vec{W} = \vec{W}(t, \vec{r}, \dot{\vec{r}}) = \vec{W}(W_r, W_\tau, W_n)$ - perturbing function.

When

$$\tilde{a} = \frac{1}{2} \left( \frac{\dot{v}}{v} + \frac{\dot{\gamma}}{\gamma} \right), \quad \tilde{b} = \frac{\ddot{\gamma}}{\gamma} - \frac{1}{2} \left( \frac{\dot{v}}{v} + \frac{\dot{\gamma}}{\gamma} \right) \frac{\dot{\gamma}}{\gamma}, \quad \vec{W} = 0, \quad (3.2)$$

where  $\gamma = \gamma(t)$  is an arbitrary dimensionless time function, the concrete form of it we determine depending on the concrete form of the equation (3.1) of the considered problem, we have a perturbed motion.

For our goals, the following system of oscillating elements is preferable [9]

$$a, e, i, \pi, \Omega, \lambda, \quad (3.3)$$

which are analogues of the semi-major axis, eccentricity, inclination, longitude of the pericenter, longitude of the ascending node, and mean longitude in the orbit of the corresponding Keplerian elements, in the case of quasi-elliptical motion ( $e(t) < 1$ ).

To investigate our restricted three-body problem with variable masses varying non-isotropically at different rates, in the presence of reactive forces in the relative coordinate system (2.6), we further use perturbed motion equations in Newton form

$$\dot{a} = \frac{2a^2 e \sin \theta}{p} \cdot \tilde{F}_r + \frac{2a^2}{r} \cdot \tilde{F}_\tau, \quad p = a(1 - e^2) \quad (3.4)$$

$$\dot{e} = \sin \theta \cdot \tilde{F}_r + \left( \cos \theta + \frac{e + \cos \theta}{1 + e \cdot \cos \theta} \right) \cdot \tilde{F}_\tau, \quad (3.5)$$

$$\frac{di}{dt} = \frac{\cos u}{1 + e \cdot \cos \theta} \cdot \tilde{F}_n, \quad (3.6)$$

$$\dot{\pi} = -\frac{\cos \theta}{e} \cdot \tilde{F}_r + \frac{\sin \theta}{e} \left( 1 + \frac{r}{\gamma p} \right) \cdot \tilde{F}_\tau + \frac{r}{\gamma p} \sin u \cdot \operatorname{tg} \frac{i}{2} \cdot \tilde{F}_n, \quad (3.7)$$

$$\dot{\Omega} = \frac{\sin u}{(1 + e \cos \theta) \sin i} \cdot \tilde{F}_n, \quad (3.8)$$

$$\dot{\lambda} = n \left( \frac{m_0}{m_{00}} \right)^2 - 2 \frac{r}{\gamma p} \sqrt{1 - e^2} \cdot \tilde{F}_r + \frac{r}{\gamma p} \sin u \cdot \operatorname{tg} \frac{i}{2} \cdot \tilde{F}_n + \frac{1}{1 + \sqrt{1 - e^2}} \left[ -\cos \theta \cdot \tilde{F}_r + \left( 1 + \frac{r}{\gamma p} \right) \sin \theta \cdot \tilde{F}_\tau \right], \quad (3.9)$$

where

$$\tilde{F}_r = W_r \cdot \frac{\sqrt{p}}{\sqrt{\mu_0}} \left( \frac{\gamma m_0}{m} \right)^{1/2}, \quad \tilde{F}_\tau = W_\tau \cdot \frac{\sqrt{p}}{\sqrt{\mu_0}} \left( \frac{\gamma m_0}{m} \right)^{1/2}, \quad \tilde{F}_n = W_n \cdot \frac{\sqrt{p}}{\sqrt{\mu_0}} \left( \frac{\gamma m_0}{m} \right)^{1/2}, \quad (3.10)$$

which were obtained earlier in the work [10].

**4. The restricted three-body problem with variable masses changing at different rates in the presence of reactive forces in the osculating elements.** Proceeding from motion equations of the problem under consideration in the relative coordinate system (2.4)-(2.6), let us rewrite them as perturbed motion equations on the basis of aperiodic motion along quasi-conic section. To simplify the perturbed motion equations, let us choose an arbitrary function  $\gamma = \gamma(t)$  in formulas (3.1)-(3.10) in the following concrete form



$$\left( \frac{\dot{v}}{\gamma} + \frac{\dot{\gamma}}{\gamma} \right) = 0, \quad \gamma = \frac{v(t_0)}{v(t)} = \gamma(t). \quad (4.1)$$

Introducing the notation  $m_0 + m_l = v_1$ , we rewrite equations (2.4) in the form

$$\ddot{\vec{r}}_1 + f v_1 \frac{\vec{r}_1}{r_1^3} - \frac{\ddot{\gamma}_1}{\gamma_1} \vec{r}_1 = \vec{W}_1, \quad \vec{W}_1 = -\frac{\ddot{\gamma}_1}{\gamma_1} \vec{r}_1 + \vec{F}_1 = \vec{W}_1(W_{1r}, W_{1r}, W_{1n}), \quad (4.2)$$

$$W_{1r} = -\frac{\ddot{\gamma}_1}{\gamma_1} r_1 + F_{1r}(t), \quad W_{1r} = F_{1r}(t), \quad W_{1n} = F_{1n}(t), \quad (4.3)$$

$$\gamma_1 = \frac{v_1(t_0)}{v_1(t)} = \gamma_1(t), \quad (4.4)$$

where  $\vec{W}_1(W_{1r}, W_{1r}, W_{1n})$  - perturbing force.

Then, the perturbed motion equations of the less massive main body, as a general two-body problem with variable masses in the presence of reactive forces, on the basis of aperiodic motion along the quasi-conic section (4.2) in the osculating elements (3.3), have the form

$$\dot{a}_1 = \frac{2a_1^2 e_1 \sin \theta_1}{p_1} \cdot W_{1r}^* + \frac{2a_1^2}{r_1} \cdot W_{1r}^*, \quad (4.5)$$

$$\dot{e}_1 = \sin \theta_1 \cdot W_{1r}^* + \left( \cos \theta_1 + \frac{e_1 + \cos \theta_1}{1 + e_1 \cos \theta_1} \right) \cdot W_{1r}^*, \quad (4.6)$$

$$\frac{di_1}{dt} = \frac{\cos u_1}{1 + e_1 \cos \theta_1} \cdot W_{1n}^*, \quad (4.7)$$

$$\dot{\tau}_1 = -\frac{\cos \theta_1}{e_1} \cdot W_{1r}^* + \frac{\sin \theta_1}{e_1} \left( 1 + \frac{r_1}{\gamma_1 p_1} \right) \cdot W_{1r}^* + \frac{r_1}{\gamma_1 p_1} \sin u_1 \cdot \operatorname{tg} \frac{i_1}{2} \cdot W_{1n}^*, \quad (4.8)$$

$$\dot{\Omega}_1 = \frac{\sin u_1}{(1 + e_1 \cos \theta_1) \sin i_1} \cdot W_{1n}^*, \quad (4.9)$$

$$\begin{aligned} \dot{\lambda}_1 = n_1 \left( \frac{m_0 + m_1}{m_{00} + m_{01}} \right)^2 - 2 \frac{r_1}{\gamma_1 p_1} \sqrt{1 - e_1^2} \cdot W_{1r}^* + \frac{r_1}{\gamma_1 p_1} \sin u_1 \cdot \operatorname{tg} \frac{i_1}{2} \cdot W_{1n}^* + \\ + \frac{1}{1 + \sqrt{1 - e_1^2}} \left[ -\cos \theta_1 \cdot W_{1r}^* + \left( 1 + \frac{r_1}{\gamma_1 p_1} \right) \sin \theta_1 \cdot W_{1r}^* \right], \end{aligned} \quad (4.10)$$

$$W_{1r}^* = W_{1r} \left( \frac{\sqrt{p_1}}{\sqrt{f v_{10}}} \left( \frac{\gamma_1 v_{10}}{v_1} \right)^{1/2} \right), \quad W_{1r}^* = W_{1r} \left( \frac{\sqrt{p_1}}{\sqrt{f v_{10}}} \left( \frac{\gamma_1 v_{10}}{v_1} \right)^{1/2} \right), \quad W_{1n}^* = W_{1n} \left( \frac{\sqrt{p_1}}{\sqrt{f v_{10}}} \left( \frac{\gamma_1 v_{10}}{v_1} \right)^{1/2} \right). \quad (4.11)$$

Now consider the perturbed motion equations of a massless body in the attraction field of two main bodies

$$\ddot{\vec{r}}_2 + f m_0 \frac{\vec{r}_2}{r_2^3} - \frac{\ddot{\gamma}_2}{\gamma_2} \vec{r}_2 = \vec{W}_2, \quad \vec{W}_2 = -\frac{\ddot{\gamma}_2}{\gamma_2} \vec{r}_2 + \vec{\Pi}_2 + \vec{F}_2 = \vec{W}_2(W_{2r}, W_{2r}, W_{2n}), \quad (4.12)$$

$$W_{2r} = -\frac{\ddot{\gamma}_2}{\gamma_2} r_2 + \Pi_{2r} + F_{2r}(t), \quad W_{2r} = \Pi_{2r} + F_{2r}(t), \quad W_{2n} = \Pi_{2n} + F_{2n}(t), \quad (4.13)$$

$$\gamma_2 = \frac{m_0(t_0)}{m_0(t)} = \gamma_2(t), \quad (4.14)$$

where  $\vec{W}_2(W_{2r}, W_{2r}, W_{2n})$  - perturbing force.

Accordingly, the motion equations of a infinitesimal mass body (4.12) in the osculating elements (3.3) have the form

$$\dot{a}_2 = \frac{2a_2^2 e_2 \sin \theta_2}{p_2} \cdot W_{2r}^* + \frac{2a_2^2}{r_2} \cdot W_{2r}^*, \tag{4.15}$$

$$\dot{e}_2 = \sin \theta_2 \cdot W_{2r}^* + \left( \cos \theta_2 + \frac{e_2 + \cos \theta_2}{1 + e_2 \cos \theta_2} \right) \cdot W_{2r}^*, \tag{4.16}$$

$$\frac{di_2}{dt} = \frac{\cos u_2}{1 + e_2 \cos \theta_2} \cdot W_{2n}^*, \tag{4.17}$$

$$\dot{\pi}_2 = -\frac{\cos \theta_2}{e_2} \cdot W_{2r}^* + \frac{\sin \theta_2}{e_2} \left( 1 + \frac{r_2}{\gamma_2 p_2} \right) \cdot W_{2r}^* + \frac{r_2}{\gamma_2 p_2} \sin u_2 \cdot \operatorname{tg} \frac{i_2}{2} \cdot W_{2n}^*, \tag{4.18}$$

$$\dot{\Omega}_2 = \frac{\sin u_2}{(1 + e_2 \cos \theta_2) \sin i_2} \cdot W_{2n}^*, \tag{4.19}$$

$$\begin{aligned} \dot{\lambda}_2 = n_2 \left( \frac{m_0}{m_{00}} \right)^2 - 2 \frac{r_2}{\gamma_2 p_2} \sqrt{1 - e_2^2} \cdot W_{2r}^* + \frac{r_2}{\gamma_2 p_2} \sin u_2 \cdot \operatorname{tg} \frac{i_2}{2} \cdot W_{2n}^* + \\ + \frac{1}{1 + \sqrt{1 - e_2^2}} \left[ -\cos \theta_2 \cdot W_{2r}^* + \left( 1 + \frac{r_2}{\gamma_2 p_2} \right) \sin \theta_2 \cdot W_{2r}^* \right], \end{aligned} \tag{4.20}$$

$$\begin{aligned} W_{2r}^* = W_{2r} \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right), \quad W_{2r}^* = W_{2r} \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right), \\ W_{2n}^* = W_{2n} \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right). \end{aligned} \tag{4.21}$$

**5. Averaging and obtaining evolution equations.** The evolution equations of a less massive body has the form

$$\dot{a}_1^{\text{sec}} = 2a_1 \cdot F_{1r}(t) \left( \frac{\sqrt{p_1}}{\sqrt{f v_{10}}} \left( \frac{\gamma_1 v_{10}}{v_1} \right)^{1/2} \right), \tag{5.1}$$

$$\dot{e}_1^{\text{sec}} = e_1 \frac{a_1}{p_1} \left( \frac{5 + e_1^2}{2} \right) \left( \frac{\sqrt{p_1}}{\sqrt{f v_{10}}} \left( \frac{\gamma_1 v_{10}}{v_1} \right)^{1/2} \right) F_{1r}(t), \tag{5.2}$$

$$\frac{di_1^{\text{sec}}}{dt} = -e_1 \cos \omega_1 \frac{3a_1}{2} \left( \frac{\sqrt{p_1}}{\sqrt{f v_{10}}} \left( \frac{\gamma_1 v_{10}}{v_1} \right)^{1/2} \right) F_{1n}(t), \tag{5.3}$$

$$\dot{\Omega}_1^{\text{sec}} = -e_1 \frac{3a_1 \sin \omega_1}{2 \sin i_1} F_{1n}(t) \left( \frac{\sqrt{p_1}}{\sqrt{f v_{10}}} \left( \frac{\gamma_1 v_{10}}{v_1} \right)^{1/2} \right), \tag{5.4}$$

$$\dot{\pi}_1^{\text{sec}} = \left( -\frac{3a_1}{2\gamma_1} \left( \ddot{\gamma}_1 + e_1 \frac{\sin \omega_1}{p_1} \cdot \operatorname{tg} \frac{i_1}{2} F_{1n}(t) \right) - \left( 1 + \frac{e_1^2}{2} \right) \frac{a_1}{\gamma_1 p_1} F_{1r}(t) + F_{1r}(t) \right) \left( \frac{\sqrt{p_1}}{\sqrt{f v_{10}}} \left( \frac{\gamma_1 v_{10}}{v_1} \right)^{1/2} \right), \tag{5.5}$$

$$\begin{aligned} \dot{\lambda}_1^{\text{sec}} = n_1 \left( \frac{m_0 + m_1}{m_{00} + m_{01}} \right)^2 &- \left[ (2 + 3e_1^2) \sqrt{1 - e_1^2} \frac{\ddot{\gamma}_1}{\gamma_1^2} \frac{a_1^2}{p_1} - \right. \\ &- \left. \left( (2 + e_1^2) \frac{a_1}{\gamma_1 p_1} \sqrt{1 - e_1^2} - \frac{e_1}{1 + \sqrt{1 - e_1^2}} \right) F_{1r}(t) - \right. \\ &- \left. \frac{3}{2} \frac{a_1 e_1}{\gamma_1} \left( \frac{\ddot{\gamma}_1}{1 + \sqrt{1 - e_1^2}} + \frac{\sin \omega_1}{p_1} \cdot \text{tg} \frac{i_1}{2} F_{1n}(t) \right) \right] \left( \frac{\sqrt{p_1}}{\sqrt{f v_{10}}} \left( \frac{\gamma_1 v_{10}}{v_1} \right)^{1/2} \right). \end{aligned} \quad (5.6)$$

Now let us consider the evolution equations of a infinitesimal mass body. It is known that to calculate the secular perturbations of the first order, the perturbations generated by each force can be calculated separately, then they should be summed up.

Therefore, the secular perturbations generated by the Newtonian gravitational force

$$\mathbf{\Pi}_2 = \text{grad}_{\mathbf{r}_2} U, \quad U = f m_1 \left( \frac{1}{r_{12}} - \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{r_1^3} \right), \quad r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}, \quad (5.7)$$

can be calculated separately. In our problem, as in the classical restricted three-body problem, the secular perturbations are generated only by the main part of the perturbative function of Newtonian gravity (5.7)

$$U = f m_1 \left( \frac{1}{r_{12}} \right) = U_{\text{prim.}} = f m_1 \left( \frac{1}{\Delta_{12}} \right), \quad \Delta_{12} = r_{12}. \quad (5.8)$$

Averaging the expressions (5.8) by  $\lambda_1$  we get

$$U_{\lambda_1}^{\text{sec}} = \tilde{U} = \frac{f m_1}{2\pi} \int_0^{2\pi} \frac{d\lambda_1}{\Delta_{12}}. \quad (5.9)$$

Then we can write

$$\tilde{\Pi}_{2r} = \frac{\partial \tilde{U}}{\partial \mathbf{e}_r}, \quad \tilde{\Pi}_{2\tau} = \frac{\partial \tilde{U}}{\partial \mathbf{e}_\tau}, \quad \tilde{\Pi}_{2n} = \frac{\partial \tilde{U}}{\partial \mathbf{e}_n}, \quad (5.10)$$

in the right side of it is necessary to calculate the derivatives along the directions of  $\mathbf{e}_r$ ,  $\mathbf{e}_\tau$ ,  $\mathbf{e}_n$  from the scalar quantity (5.9).

In the adopted notations (5.9), (5.10), the equations of evolution equations for the analog of the semi-major axis looks like

$$\begin{aligned} \dot{a}_2^{\text{sec}} = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{2a_2^2 e_2}{p_2} \sin \theta_2 \left[ -\frac{\ddot{\gamma}_2}{\gamma_2} r_2 + F_{2r}(t) \right] \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right) + \right. \\ \left. + \frac{2a_2^2}{r_2} F_{2\tau}(t) \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right) \right\} d\lambda_2 + \\ + \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right) \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{2}{n_2 a_2} \left( \frac{n_2 a_2^2 e_2}{(1 - e_2^2)} \sin \theta_2 [\tilde{\Pi}_{2r}] + \frac{n_2 a_2^3}{r_2} [\tilde{\Pi}_{2\tau}] \right) \right\} d\lambda_2. \end{aligned} \quad (5.11)$$

The expression (5.9) can be interpreted, according to the Gaussian interpretation, as the potential of a quasi-elliptic spiral with parameters  $(a_1, e_1, i_1, \Omega_1, \pi_1)$  and with mass  $m_1$  per outer point  $T_2$ .

Let us show that in equation (5.11) the expression in the curly bracket in the second summand is zero.

It is true that in deriving the Lagrange perturbed motion equations from Newton's equation, for secular perturbations generated by the Newtonian gravitational force, the well-known relation takes place

$$\frac{\partial U}{\partial \lambda_2} = + \frac{n_2 a_2^2 e_2 \sin \theta_2}{1 - e^2} \cdot \Pi_{2r} + \frac{n_2 a_2^3}{r_2} \cdot \Pi_{2r}. \tag{5.12}$$

In the left part of equation (5.12), after averaging over  $\lambda_2$ , it is clear that

$$\frac{\partial \tilde{U}}{\partial \lambda_2} = 0. \tag{5.13}$$

Therefore, at calculating the first-order secular perturbations, in the perturbed motion equations in the form of Newton's equations (5.11), it takes place

$$\frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{n_2 a_2^2 e_2 \sin \theta_2}{1 - e^2} \cdot \tilde{\Pi}_{2r} + \frac{n_2 a_2^3}{r_2} \cdot \tilde{\Pi}_{2r} \right\} d\lambda_2 = 0. \tag{5.14}$$

Therefore, the second component in equation (5.11) is zero

$$\left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right) n_2 a_2 \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{n_2 a_2^2 e_2}{(1 - e^2)} \sin \theta_2 [\Pi_{2r}] + \frac{n_2 a_2^3}{r_2} [\Pi_{2r}] \right] d\lambda_2 \right\} = 0.$$

Thus, equation (5.11) has the form

$$\begin{aligned} \dot{a}_2^{\text{sec}} = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{2a_2^2 e_2}{p_2} \sin \theta_2 \left( -\frac{\ddot{\gamma}_2}{\gamma_2} r_2 + F_{2r}(t) \right) \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right) + \right. \\ \left. + \frac{2a_2^2}{r_2} F_{2r}(t) \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right) \right\} d\lambda_2. \end{aligned} \tag{5.15}$$

Expanding the integrals into series according to the formulas of unperturbed motion up to the second degree of eccentricity, and averaging by  $\lambda_2$  from equation (5.15) we get

$$\dot{a}_2^{\text{sec}} = 2F_{2r}(t) \left( 1 + \frac{e_2^2}{2} \right) \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right). \tag{5.16}$$

By averaging, we obtain evolutionary equations for the other osculating elements

$$\begin{aligned} \dot{e}_2^{\text{sec}} = \left[ -a_2 \frac{\ddot{\gamma}_2}{\gamma_2} - e_2 \left( \tilde{\Pi}_{2r}^{\text{sec}} + F_{2r}(t) \right) + e_2 \frac{a_2}{p_2} \left( 1 + \frac{e_2^2}{2} \right) \left( \tilde{\Pi}_{2r}^{\text{sec}} + F_{2r}(t) \right) - \right. \\ \left. - e_2 \frac{3a_2}{2p_2} \left( \tilde{\Pi}_{2r}^{\text{sec}} - F_{2r}(t) \right) \right] \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right), \end{aligned} \tag{5.17}$$

$$\frac{di_2^{\text{sec}}}{dt} = -\frac{3}{2} \frac{a_2}{p_2} e_2 \cos \omega_2 \left( \tilde{\Pi}_{2n}^{\text{sec}} + F_{2n}(t) \right) \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right), \tag{5.18}$$

$$\dot{\Omega}_2^{\text{sec}} = -\frac{3}{2} \frac{a_2}{p_2} \frac{\sin \omega_2}{\sin i_2} e_2 \left( \tilde{\Pi}_{2n}^{\text{sec}} + F_{2n}(t) \right) \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right), \tag{5.19}$$

$$\begin{aligned} \dot{\kappa}_2^{\text{sec}} = \left[ -a_2 \frac{3}{2} \left( \frac{\ddot{\gamma}_2}{\gamma_2} + \frac{e_2}{\gamma_2 p_2} \text{tg} \frac{i_2}{2} \sin \omega_2 \cdot F_{2n}(t) \right) - \tilde{\Pi}_{2r}^{\text{sec}} - \right. \\ \left. - F_{2r}(t) - \frac{a_2 e_2}{\gamma_2 p_2} \text{tg} \frac{i_2}{2} \sin \omega_2 \frac{3}{2} \tilde{\Pi}_{2n}^{\text{sec}} \right] \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right), \end{aligned} \tag{5.20}$$

$$\begin{aligned}
 \dot{\lambda}_2^{\text{sec}} = & n_2 \left( \frac{m_0}{m_{00}} \right)^2 + \left[ \frac{2a_2^2 \ddot{\gamma}_2}{p_2 \gamma_2^2} \left( 1 + \frac{3}{2} e_2^2 \right) \sqrt{1 - e_2^2} - \right. \\
 & - \left( 1 + \frac{e_2^2}{2} \right) \frac{2a_2}{\gamma_2 p_2} \sqrt{1 - e_2^2} (1 + F_{2r}(t)) - \frac{a_2 e_2}{\gamma_2 p_2} \cdot \text{tg} \frac{i_2}{2} \cdot \sin \omega_2 \frac{3}{2} \tilde{\Pi}_{2n}^{\text{sec}} + \\
 & + \frac{a_2}{\gamma_2 p_2} \left( \text{tg} \frac{i_2}{2} \cdot F_{2n}(t) \cos \omega_2 + \frac{1}{1 + \sqrt{1 - e_2^2}} F_{2r}(t) \right) - \\
 & - \frac{3e_2 a_2}{2\gamma_2} \left( \frac{\sin \omega_2}{p_2} \cdot \text{tg} \frac{i_2}{2} \cdot F_{2n}(t) + \frac{\ddot{\gamma}_2}{1 + \sqrt{1 - e_2^2}} \right) + \\
 & \left. + \frac{e_2}{1 + \sqrt{1 - e_2^2}} \left( \tilde{\Pi}_{2r}^{\text{sec}} + F_{2r}(t) \right) \right] \left( \frac{\sqrt{p_2}}{\sqrt{f m_{00}}} \left( \frac{\gamma_2 m_{00}}{m_0} \right)^{1/2} \right),
 \end{aligned} \tag{5.21}$$

where

$$\frac{1}{2\pi} \int_0^{2\pi} \tilde{\Pi}_{2r} d\lambda_2 = \tilde{\Pi}_{2r}^{\text{sec}}, \quad \frac{1}{2\pi} \int_0^{2\pi} \tilde{\Pi}_{2n} d\lambda_2 = \tilde{\Pi}_{2n}^{\text{sec}}, \quad \frac{1}{2\pi} \int_0^{2\pi} \tilde{\Pi}_{2r} d\lambda_2 = \tilde{\Pi}_{2r}^{\text{sec}}. \tag{5.22}$$

**Conclusion.** We obtained the evolution equations of motion of a less massive body in osculating elements. Evolution equations of a infinitesimal mass body in osculating elements are obtained in principle, in fact it is necessary, with the necessary exactitude, to give an expansion of the Newtonian force function. Such expansion of the perturbative function of the Newtonian gravitational force will be obtained using the Wolfram Mathematica package. The evolution equations are suitable for the general case, i.e. when the masses of bodies change non-isotropically at different rates (both mass growth and mass decrease), in the presence of reactive forces.

After the expansion of the perturbative function generated by the Newtonian gravitational force, we further investigate it numerically, using Wolfram Mathematica mathematical package. The resulting equations will be suitable not only for planetary systems, but also for binary stars.

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### МАССАЛАРЫ АЙНЫМАЛЫ ШЕКТЕЛГЕН ҮШ ДЕНЕ МӘСЕЛЕСІНІҢ ЭВОЛЮЦИЯЛЫҚ ТЕНДЕУЛЕРІ

**Аннотация.** Классикалық аспан механикасында кеплер қоғалысы негізінде ұйытқу теориясы жақсы дамыған. Ұйытқыған қозғалыс теңдеулерінің түрлері конондық теңдеулермен, Лагранж және Ньютон теңдеулерімен сипатталған. Сонымен қатар массалары тұрақты денелердің, нүкте ретінде немесе сфералық дене ретінде қарастырылады. Көптеген астрономиялық мәселелерді зерттеуде классикалық ұйытқу теориясы басты құрал болып табылады. Заманауи бақылау астрономиясы нақты ғарыштық денелердің массалары уақыт өте өзгертінін растауда. Шынайы ғарыштық денелер сфералық емес және қатты дене емес. Аспан денелері бейстационар, эволюция кезеңінде олардың массалары, өлшемдері, пішіндері және құрылымы өзгереді. Қазіргі таңда айнымалы массалы аспан механикасы бірқатар бойынша ауқымды зерттеулер библиографиялық жұмыстары бар. Алайда,

бейстационар ғарыштық жүйелер үшін ұйытқу теориясы жеткілікті дамымаған. Осыған орай біз шектелген қойылымдағы массалары айнымалы, тығыздығы сфера бойынша үлестірілген гравитацияланушы үш сфералы аспан денелерінен тұратын жүйені қарастырдық. Сонымен қатар, біз реактивті күші бар денелердің массалары изотропты емес әр түрлі қарқында өзгертін жалпы жағдайды қарастырдық. Массасы аз дене екі негізгі денелердің қозғалысына әсер етпейді деп есептейміз. Аталған мәселе бейстационар гравитацияланатын жүйелер үшін даярланған квазиконустық қима бойынша аperiodты қозғалыс негізіндегі ұйытқыған теория әдістерімен зерттелді. Салыстырмалы координат жүйесіндегі массивті дене мен массасы аз дененің оскуляциялаушы элементтерінің эволюциялық теңдеулері алынды.

**Түйін сөздер:** шектелген үш дене мәселесі, массаның изотропты емес өзгеруі, реактивті күштер, квазиконустық қима бойынша аperiodты қозғалыс, Ньютон теңдеулері түріндегі ұйытқыған қозғалыс теңдеулері.

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### **ЭВОЛЮЦИОННЫЕ УРАВНЕНИЯ ОГРАНИЧЕННОЙ ЗАДАЧИ ТРЕХ ТЕЛ С ПЕРЕМЕННЫМИ МАССАМИ**

**Аннотация.** В классической небесной механике хорошо развита теория возмущения на базе кеплеровского движения. Различные виды уравнения возмущенного движения описаны каноническими уравнениями, уравнениями Лагранжа и уравнениями Ньютона. При этом массы тел считаются постоянными, тела точечными или сферическими. Классическая теория возмущения является основным инструментом в изучении многих астрономических проблем. Современная наблюдательная астрономия свидетельствует о том, что массы реальных космических тел со временем меняются. Реальные небесные тела не сферичны и не твердые. Небесные тела нестационарные, в процессе эволюции меняются их массы, размеры, формы и структуры. В настоящее время существует ряд исследований с обширной библиографией работ по небесной механике тел переменной массы. Однако теория возмущения для нестационарных космических систем не достаточно развита. В связи с этим мы рассмотрели гравитирующую систему состоящую из трех сферических небесных тел, со сферическими распределениями плотностей, с переменными массами в ограниченной постановке. Мы рассмотрели общий случай, когда массы тел изменяются не изотропно в различных темпах, при наличии реактивных сил. Будем считать, что тело с бесконечно малой массой не влияет на движения двух основных тел. Проблема исследована методами теории возмущения на основе аperiodического движения по квазиконическому сечению, разработанных для нестационарных гравитирующих систем. Получены эволюционные уравнения движения менее массивного тела и малого тела в оскулирующих элементах, в относительной системе координат.

**Ключевые слова:** ограниченная задача трех тел, не изотропное изменения масс, реактивные силы, аperiodическое движение по квазиконическому сечению, уравнения возмущенного движения в форме уравнения Ньютона.

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#### REFERENCE

- [1] Minglibayev M.Zh., Ibraimova A.T. Equations of motion of the restricted three-body problem with non-isotropically variable masses with reactive forces // *News of the National Academy of Sciences of the Republic of Kazakhstan. Physico-mathematical series.* – 2019. – T.325, №3. – C. 5-12. <https://doi.org/10.32014/2019.2518-1726.18>
- [2] Omarov T.B. *Dinamika gravitiruyushchikh sistem metagalaktiki* [Dynamics of gravitating systems of the metagalaxy] // *Izd. Nauka, Kazakhskoy SSR. Alma-Ata.* – 1975. -144 p. (in Russ.).
- [3] Omarov T.B. (Editor) *Non-Stationary Dynamical Problems in Astronomy.* New-York: Nova Science Publ.Inc. – 2002. - 260 p.
- [4] Bekov A.A., Omarov T.B. The theory of orbits in non-stationary stellar systems // *Astron. Astrophys. Trans.* – 2003. – V. 22, №2. – P.145-153.
- [5] Eggleton P. *Evolutionary processes in binary and multiple stars.* Cambridge University Press. – 2006. – 332 p.
- [6] Veras D., Post-main-sequence planetary system evolution, *Royal Soc. Open Science*, 2016. DOI: 10.1098/rsos.150571
- [7] Minglibayev M.Dzh., *Dinamika gravitiruyushchikh tel s peremennymi massami i razmerami* [Dynamics of gravitating bodies with variable masses and sizes]. LAP LAMBERT Academic Publishing. 2012. 224 p. (in Russ.).
- [8] Cherepashchuk A.M. *Tesnyye dvoynnye zvezdy* [Close binary stars]. Chast' II.-M.: Fizmatlit, 2013.- 572 p.
- [9] Grebenikov Ye.A., Ryabov Yu.A. *Novyye kachestvennyye metody v nebesnoy mekhanike* [New qualitative methods in celestial mechanics]. Nauka. - 1971. - 444 p.
- [10] Minglibayev M.Zh., Omarov Ch.T., Ibraimova A.T. New forms of the perturbed motion equation // *Reports of the national academy of sciences of the republic of Kazakhstan.* – 2020. – V.2 (330). – P.5-13. <https://doi.org/10.32014/2020.2518-1483.25>

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