ISSN 1991-346X

ҚАЗАҚСТАН РЕСПУБЛИКАСЫ
ҰЛТТЫҚ ҒЫЛЫМ АКАДЕМИЯСЫНЫҢ

Х А Б А Р Л А Р Ы

ИЗВЕСТИЯ

НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК
РЕСПУБЛИКИ КАЗАХСТАН

N E W S

OF THE NATIONAL ACADEMY OF SCIENCES
OF THE REPUBLIC OF KAZAKHSTAN

ФИЗИКА-МАТЕМАТИКА СЕРИЯСЫ

♦

СЕРИЯ ФИЗИКО-МАТЕМАТИЧЕСКАЯ

♦

PHYSICO-MATHEMATICAL SERIES

3 (301)

МАМЫР – МАУСЫМ 2015 ж.
МАЙ – ИЮНЬ 2015 г.
MAY – JUNE 2015

1963 ЖЫЛДЫҢ ҚАҢТАР АЙЫНАН ШЫҒА БАСТАҒАН
ИЗДАЕТСЯ С ЯНВАРЯ 1963 ГОДА
PUBLISHED SINCE JANUARY 1963

ЖЫЛЫНА 6 РЕТ ШЫҒАДЫ
ВЫХОДИТ 6 РАЗ В ГОД
PUBLISHED 6 TIMES A YEAR

АЛМАТЫ, ҚР ҰҒА
АЛМАТЫ, НАН РК
ALMATY, NAS RK

«Известия НАН РК. Серия физико-математическая». ISSN 1991-346X
Собственник: РОО «Национальная академия наук Республики Казахстан» (г. Алматы)
Свидетельство о постановке на учет периодического печатного издания в Комитете информации и архивов Министерства культуры и информации Республики Казахстан **№5543-Ж**, выданное 01.06.2006 г.

Периодичность: 6 раз в год.
Тираж: 300 экземпляров.

Адрес редакции: 050010, г. Алматы, ул. Шевченко, 28, ком. 219, 220, тел.: 272-13-19, 272-13-18,
www:nauka-nanrk.kz / physics-mathematics.kz

Адрес типографии: ИП «Аруна», г. Алматы, ул. Муратбаева, 75.

News of the National Academy of Sciences of the Republic of Kazakhstan. Physical-mathematical series.
ISSN 1991-346X
Owner: RPA "National Academy of Sciences of the Republic of Kazakhstan" (Almaty)
The certificate of registration of a periodic printed publication in the Committee of information and archives of the Ministry of culture and information of the Republic of Kazakhstan N 5543-Ж, issued 01.06.2006

Periodicity: 6 times a year
Circulation: 300 copies

Editorial address: 28, Shevchenko str., of. 219, 220, Almaty, 050010, tel. 272-13-19, 272-13-18,
www:nauka-nanrk.kz / physics-mathematics.kz

Address of printing house: ST "Aruna", 75, Muratbayev str, Almaty

N E W S
OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X
Volume 3, Number 301 (2015), 111 – 115

SUBADDITIVITY WEAK MAJORIZATION INEQUALITIES FOR τ-MEASURABLE OPERATORS

D. Dostilek

Institute of mathematics and mathematical modeling NAS RK,
Al-Farabi Kazakh National University, Almaty, Kazakhstan.
E-mail: dostilek.dauitbek@gmail.com

Key words: subadditivity inequality, τ-measurable operator, expansive operator, von Neumann algebra, submajorization.

Abstract. Let (M,τ) be a semi-finite von Neumann algebra and let and $f:[0,\infty)\to[0,\infty)$ is increasing concave function such that $f(0)=0$ We have the following results:

- If x, y are τ-measurable operators, then $f(|x+y|)\preceq f(|x|)+f(|y|)$.
- If x is self-adjoint τ-measurable operator and $z\in M$ is expansive operator, then $f(|z^*xz|)\preceq z^*f(|x|)z$.

1. **Introduction.** Let M_n be von Neumann algebra of $n\times n$ complex matrices, and let M_n^+ be positive part of M_n. Bourin in [3] proved the following a matrix subadditivity inequality for symmetric norms: Let $f:[0,\infty)\to[0,\infty)$ be concave.

- If A and B be normal matrices. Then, for all symmetric norms

$$\|f(|A+B|)\|\le\|f(|A|)+f(|B|)\|.$$

- If A be normal and let Z be expansive. Then, for all symmetric norms

$$\|f(|Z^*AZ|)\|\le\|Z^*f(|A|)Z\|.$$

The purpose of this paper is to extend the above results to n-tuples of τ-measurable operators.

This paper is organized as follows. Section 2 contains some preliminary definitions.

In section 3, we proved the weak majorization type of subadditivity inequalities for n tuples of τ-measurable operators.

2. Preliminaries. Throughout this paper, we denote by M a semi-finite von Neumann algebra on the Hilbert space H with a normalized normal faithful finite trace τ. The closed densely defined linear operator x in H with domain $D(x)$ is said to be affiliated with M if and only if $u^*xu=x$ for all unitary u which belong to the commutant M' of M. If x is affiliated with M, the x said to be τ-measurable if for every $\varepsilon>0$ there exists a projection $e\in M$ such that $e(H)\subseteq D(x)$ and $\tau(e^\perp)<\varepsilon$ (where for any projection e we let $e^\perp=1-e$). The set of all τ-measure operators will be denoted by $L_0(M)$. The set $L_0(M)$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product. Let $P(M)$ be the lattice of projections of M. The sets

$$N(\varepsilon,\delta)=\{x\in L_0(M):\exists\, e\in P(M)\text{ such that }\|xe\|<\varepsilon,\tau(e^\perp)<\delta\}$$

$(\varepsilon,\delta>0)$ from a base at 0 for an metrizable Hausdorff topology in $L_0(M)$ called the measure topology. Equipped with the measure topology, $L_0(M)$ is a complete topological $*$-algebra (see [6]). For $x\in L_0(M)$, the generalized singular value function $\mu(x)$ of x is defined by

$$\mu_s(x)=\inf\{\|xe\|;e\in P(M)\,\tau(e^\perp)\le s\},\ s\ge 0.$$

If $x,y\in L_0(M)$, then we say that x is submajorized by y and write $x\preceq y$ if and only if

$$\int_0^t\mu_s(x)ds\le\int_0^t\mu_s(y)ds,\quad t\in[0,1].$$

The following lemma is a well-known result (see [4], Theorem 5.2.).

Lemma 21.1. *Let x,y be positive τ-measurable operators and $f:[0,\infty)\to[0,\infty)$ be a concave function. Then*

$$f(x+y)\preceq f(x)+f(y). \tag{1}$$

In particular, Theorem 3.2.8., In [2], we obtain the following result.

Lemma 2.2. *Let $f:[0,\infty)\to[0,\infty)$ be a concave function, and let $z\in M$ be an expansive operator. Then*

$$f(z^*xz)\preceq z^*f(x)z,\quad\forall\, x\in L_0(M)^+. \tag{2}$$

3. Main result

Theorem 3.12 *Let* x, y *be τ-measurable operators and let and* $f:[0,\infty)\to[0,\infty)$ *is increasing concave function such that* $f(0)=0$. *Then*

$$f(|x+y|)\preceq f(|x|)+f(|y|). \tag{3}$$

Proof. Lemma 4.3 in [5] there exist partial isometries u, v in M such that

$$|x+y|\preceq u|x|u^*+v|y|v^*$$

Applying Lemma 2.5. (iii) in [5].

$$\mu_s(|x+y|)\le \mu_s(u|x|u^*+v|y|v^*)$$

Since continuous increasing of f , we have that

$$f(\mu_s(|x+y|))\le f(\mu_s(u|x|u^*+v|y|v^*)).$$

By Lemma 2.5. (iv) in [5], we get

$$\mu_s(f(|x+y|))\le \mu_s(f(u|x|u^*+v|y|v^*)).$$

Using inequality (1), we obtain that

$$f(|x+y|)\preceq f(u|x|u^*+v|y|v^*)\preceq f(u|x|u^*)+f(v|y|v^*).$$

By Lemma 2.5. (iii) in [5], we take

$$\mu_s(u\,|\,x\,|\,u^*)\le \|u\|\|u^*\|\mu_s(|\,x\,|)\le \mu_s(|\,x\,|)\,.$$

Since continuous increasing of f

$$f(\mu_s(u\,|\,x\,|\,u^*))\le f(\mu_s(|\,x\,|))\,.$$

By Lemma 2.5. (iv) in [5], we have that

$$\mu_s(f(u\,|\,x\,|\,u^*))\le \mu_s(f(|\,x\,|))$$

so

$$f(u|x|u^*)\preceq f(|x|)$$

and we obtain the same inequality as a following:

$$f(v|y|v^*)\preceq f(|y|).$$

Hence

$$f(|x+y|)\preceq f(|x|)+f(|y|).$$

Corollary 3.1. 3 *Let* x, y *be normal τ-measurable operators and let and* $f:[0,\infty)\to[0,\infty)$ *is increasing concave function such that* $f(0)=0$. *Then*

$$f(|x+y|)\preceq f(|x|)+f(|y|).$$

Theorem 3.2. 4 *Let* x *be self-adjoint* $\tau-$*measurable operators and let and* $f:[0,\infty)\to[0,\infty)$ *is increasing concave function such that* $f(0)=0$. *Then*

$$f(|z^*xz|)\preceq z^*f(|x|)z. \tag{4}$$

Proof. It is clear that

$$x\le|\,x\,|.$$

Then by using Proposition 4.5 (iii) [7]

$$z^* xz \le z^* |x| z .$$

By Lemma 2.5. (ii),(iii) in [5], we take

$$\mu_s(|z^* xz|) = \mu_s(z^* xz) \le \mu_s(z^* |x| z) .$$

Since continuous increasing of f, we get

$$f(\mu_s(|z^* xz|)) \le f(\mu_s(z^* |x| z)) .$$

Applying Lemma 2.5. (iv) in [5], we have that

$$\mu_s(f(|z^* xz|)) \le \mu_s(f(z^* |x| z)) .$$

By continuously of integral and inequality (2), we obtain

$$f(|z^* xz|) \preceq f(z^* |x| z) \preceq z^* f(|x|) z .$$

Theorem 5 3.3. *Let* x *be τ-measurable operators and let and* $f:[0,\infty)\to[0,\infty)$ *is increasing concave function such that* $f(0)=0$. *Then*

$$\begin{pmatrix} f(|z^* xz|) & 0 \\ 0 & f(|z^* x^* z|) \end{pmatrix} \preceq \begin{pmatrix} z^* f(|x|) z & 0 \\ 0 & z^* f(|x^*|) z \end{pmatrix}. \tag{5}$$

Proof. Applying Theorem 3.2 to the Hermitian operators

$$\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}$$

we obtain

$$f\left(\left|\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}^* \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right|\right) \preceq \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}^* f\left(\left|\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\right|\right) \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

Hence

$$\begin{pmatrix} f(|z^* xz|) & 0 \\ 0 & f(|z^* x^* z|) \end{pmatrix}$$

$$= f(\begin{pmatrix} |z^* xz| & 0 \\ 0 & |z^* x^* z| \end{pmatrix}) = f\left(\left|\begin{pmatrix} 0 & z^* x^* z \\ z^* xz & 0 \end{pmatrix}\right|\right)$$

$$= f\left(\left|\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}^* \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right|\right)$$

$$\preceq \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}^* f\left(\left|\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}\right|\right) \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

$$= \begin{pmatrix} z^* & 0 \\ 0 & z^* \end{pmatrix} f(\begin{pmatrix} |x| & 0 \\ 0 & |x^*| \end{pmatrix}) \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

$$= \begin{pmatrix} z^* & 0 \\ 0 & z^* \end{pmatrix} \begin{pmatrix} f(|x|) & 0 \\ 0 & f(|x^*|) \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

$$= \begin{pmatrix} z^* f(|x|) z & 0 \\ 0 & z^* f(|x^*|) z \end{pmatrix}$$

REFERENCES

[1] Bhatia R. *Matrix analysis, Springer-Verlag*, New York, 1997.
[2] Turdebek N.B., Dostilek D., Submajorization *inequalities of τ-measurable operators for concave and convex functions* **DOI: 10.1007/s11117-014-0300-x**.
[3] Bourin J.-C. *A matrix subadditivity inequality for symmetric norms*, Proc. AMS 138 (2010), 495–504.
[4] Dodds P.G., Sukochev F.A. *Submajorisation inequalities for convex and concave functions of sums of measurable operators*, Birkhauser Verlag Basel **13** (2009), 107–124.
[5] Fack T., Kosaki H. *Generalized* S *-numbers of τ-measure operators*, Pac. J. Math. **123** (1986), 269-300.
[6] Nelson E. *Notes on non-Commutative integration*, J. Funct. Anal. **15** (1974) 103-116.
[7] Ben de Pagter. *Non-commutative Banach function spaces, Positivity, Trends Math.*, Birkhäuser, Basel, (2007), 197–227.
[8] Pisier G., Xu Q. *Noncommutative* L^p *-spaces*, Handbook of the geometry of Banach spaces*, vol.2*: 1459-1517, 2003.

τ-ӨЛШЕМДІ ОПЕРАТОРЛАР ҮШІН СУБАДДИТИВТІ ӘЛСІЗ МАЖОРЛАНҒАН ТЕҢСІЗДІКТЕР

Д. Достілек

ҚР ҰҒА Математика және математикалық моделдеу институты,
Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан

Тірек сөздер: субаддитивті теңсіздіктері, *τ*-өлшемді оператор, кеңейтуші оператор, Фон Нейман алгебрасы, субмажорланған.

Аңдатпа. (M,τ) жартылай ақырлы фон Нейман алгебрасы және $f:[0,\infty)\to[0,\infty)$ $f(0)=0$ болатын өспелі ойыс функция болсын. Біз келесі нәтижелерді аламыз.

(1) Егер *x*, *y* тер *τ*-өлшемді операторлар болса, онда

$$f(|x+y|)\preceq f(|x|)+f(|y|).$$

(2) Егер *x* өзіне-өзі түйіндес *τ*-өлшемді оператор және $z\in M$ кеңейтуші оператор болса, онда

$$f(|z^*xz|)\preceq z^*f(|x|).$$

СУБАДДИТИВНОСТЬ СЛАБО МАЖОРИЗАЦИОННЫХ НЕРАВЕНСТВ ДЛЯ *τ*-ИЗМЕРИМЫХ ОПЕРАТОРОВ

Д. Достилек

Институт математики и математического моделирования МОН РК,
Казахский национальный университет им. аль-Фараби, Алматы, Казахстан

Ключевые слова: субаддитивное неравенство, *τ*-измеримый оператор, расширяющий оператор, алгебра Фон Неймана, субмажоризация.

Аннотация. Пусть (M,τ) полу конечная алгебра Фон Неймана и $f:[0,\infty)\to[0,\infty)$ возрастающая вогнутая функция с $f(0)=0$. Мы получили следущие резултаты.

(1) Если *x*, *y* *τ*-измеримые операторы, то

$$f(|x+y|)\preceq f(|x|)+f(|y|).$$

(2) Если *x* самосопряженный *τ*-измеримый оператор и $z\in M$ расширяющий оператор, то

$$f(|z^*xz|)\preceq z^*f(|x|).$$

Поступила 25.02.2015 г.

Правила оформления статьи для публикации в журнале смотреть на сайте:

www:nauka-nanrk.kz

physics-mathematics.kz

Редактор *М. С. Ахметова*
Верстка на компьютере *Д. Н. Калкабековой*

Подписано в печать 9.06.2015.
Формат 60х881/8. Бумага офсетная. Печать – ризограф.
15,7 п.л. Тираж 300. Заказ 3.

Национальная академия наук РК
050010, Алматы, ул. Шевченко, 28, т. 272-13-18, 272-13-19