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# Х А Б А Р Л А Р Ы

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НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК  
РЕСПУБЛИКИ КАЗАХСТАН

## NEWS

OF THE NATIONAL ACADEMY OF SCIENCES  
OF THE REPUBLIC OF KAZAKHSTAN

**ФИЗИКА-МАТЕМАТИКА  
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## CLARKSON WEAK MAJORIZATION INEQUALITIES FOR $\tau$ -MEASURABLE OPERATORS

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**Key words:** Clarkson inequality,  $\tau$ -measurable operator, von Neumann algebra, submajorization.

**Abstract.** Let  $(M, \tau)$  be a semi-finite von Neumann algebra and  $f$  be a nonnegative function on  $[0, \infty)$  with  $f(0) = 0$ . Let  $x_1, x_2, \dots, x_n$  be  $\tau$ -measurable operators and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be positive real numbers such

that  $\sum_{j=1}^n \alpha_j = 1$ . We have the following results.

(1) If  $g(t) = f(\sqrt{t})$  is convex, then

$$f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right) \leq \sum_{j=1}^n \alpha_j f(|x_j|).$$

(2) If  $h(t) = f(\sqrt{t})$  is concave, then

$$\sum_{j=1}^n \alpha_j f(|x_j|) \leq f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right).$$

**1. Introduction.** Let  $M_n$  be von Neumann algebra of  $n \times n$  complex matrices, and let  $M_n^+$  be positive part of  $M_n$ . Hirzallah and Kittaneh in [5] proved the following noncommutative Clarkson inequalities for  $n$ -tuples of operators: Let  $\|\cdot\|$  be a unitarily norm,  $A_0, \dots, A_{n-1} \in M_n^+$  and  $\alpha_0, \dots, \alpha_{n-1}$  be positive real numbers such that  $\sum_{j=0}^{n-1} \alpha_j = 1$ .

(1) If  $f$  be a nonnegative function on  $[0, \infty)$  such that  $f(0) = 0$  and  $g(t) = f(\sqrt{t})$  is convex on  $[0, \infty)$ , then

$$\left\| \left\| f\left(\sum_{j=0}^{n-1} \alpha_j A_j\right) + \sum_{0 \leq j < k \leq n-1} f\left(\sqrt{\alpha_j \alpha_k} |A_j - A_k|\right) \right\| \right\| \leq \left\| \sum_{j=0}^{n-1} \alpha_j f(|A_j|) \right\|.$$

(2) If  $f$  be a nonnegative function on  $[0, \infty)$  such that  $g(t) = f(\sqrt{t})$  is concave on  $[0, \infty)$ , then

$$\left\| \sum_{j=0}^{n-1} \alpha_j f(|A_j|) \right\| \leq \left\| \left\| f\left(\sum_{j=0}^{n-1} \alpha_j A_j\right) + \sum_{0 \leq j < k \leq n-1} f\left(\sqrt{\alpha_j \alpha_k} |A_j - A_k|\right) \right\| \right\|.$$

The purpose of this paper is to extend the above results to  $n$ -tuples of  $\tau$ -measurable operators.

This paper is organized as follows. Section 2 contains some preliminary definitions.

In section 3, we proved the weak majorization type of Clarkson inequalities for  $n$  tuples of  $\tau$ -measurable operators.

**2. Preliminaries.** Throughout this paper, we denote by  $M$  a semi-finite von Neumann algebra on the Hilbert space  $H$  with a normalized normal faithful finite trace  $\tau$ . The closed densely defined linear operator  $x$  in  $H$  with domain  $D(x)$  is said to be affiliated with  $M$  if and only if  $u^* x u = x$  for all unitary  $u$  which belong to the commutant  $M'$  of  $M$ . If  $x$  is affiliated with  $M$ , the  $x$  said to be  $\tau$ -measurable if for every  $\varepsilon > 0$  there exists a projection  $e \in M$  such that  $e(H) \subseteq D(x)$  and  $\tau(e^\perp) < \varepsilon$  (where for any projection  $e$  we let  $e^\perp = 1 - e$ ). The set of all  $\tau$ -measure operators will be denoted by  $L_0(M)$ . The set  $L_0(M)$  is a  $*$ -algebra with sum and product being the respective closure of the algebraic sum and product. Let  $P(M)$  be the lattice of projections of  $M$ . The sets

$$\mathfrak{N}(\varepsilon, \delta) = \left\{ x \in L_0(M) : \exists e \in P(M) \text{ such that } \|xe\| < \varepsilon, \tau(e^\perp) < \delta \right\}$$

$(\varepsilon, \delta > 0)$  form a base at 0 for an metrizable Hausdorff topology in  $L_0(M)$  called the measure topology. Equipped with the measure topology,  $L_0(M)$  is a complete topological  $*$ -algebra (see [6]). For  $x \in L_0(M)$ , the generalized singular value function  $\mu(x)$  of  $x$  is defined by

$$\mu_s(x) = \inf \left\{ \|xe\| : e \in P(M), \tau(e^\perp) \leq s \right\} \quad (s \geq 0).$$

If  $x, y \in L_0(M)$ , then we say that  $x$  is submajorized by  $y$  and write  $x \preceq y$  if and only if

$$\int_0^t \mu_s(x) ds \leq \int_0^t \mu_s(y) ds, \quad \forall t > 0.$$

As Proposition 4.6 in [3], we obtain the following result.

**Lemma 2.1.** Let  $f$  be a continuous increasing function on  $\mathbf{R}_+$  with  $f(0) = 0$ . Let  $x_1, x_2, \dots, x_n$  be positive  $\tau$ -measurable operators and let  $a_1, a_2, \dots, a_n$  be positive elements in  $M$  with

$$\sum_{j=1}^n a_j^* a_j \leq 1.$$

(1) When  $f$  is convex, we have

$$\mu_s \left[ f \left( \sum_{j=1}^n a_j^* x_j a_j \right) \right] \leq \mu_s \left[ \sum_{j=1}^n a_j^* f(x_j) a_j \right], \quad s > 0. \quad (2.1)$$

(2) When  $f$  is concave, we have

$$\mu_s \left[ \sum_{j=1}^n a_j^* f(x_j) a_j \right] \leq \mu_s \left[ f \left( \sum_{j=1}^n a_j^* x_j a_j \right) \right], \quad s > 0. \quad (2.2)$$

The following lemma is a well-known result (see [2], Theorem 5.3.).

**Lemma 2.2.** Let  $x_1, x_2, \dots, x_n$  be positive  $\tau$ -measurable operators.

(1) If  $g: [0, \infty) \rightarrow [0, \infty)$  be a convex function with  $g(0) = 0$ . Then

$$\sum_{j=1}^n g(x_j) \preceq g \left( \sum_{j=1}^n x_j \right). \quad (2.3)$$

(2) If  $h: [0, \infty) \rightarrow [0, \infty)$  be a concave function. Then

$$h \left( \sum_{j=1}^n x_j \right) \preceq \sum_{j=1}^n h(x_j). \quad (2.4)$$

**3. Main result.** Let  $x_1, x_2, \dots, x_n$  be positive  $\tau$ -measurable operators and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be positive real numbers such that  $\sum_{j=1}^n \alpha_j = 1$ . Then

$$\left| \sum_{j=1}^n \alpha_j x_j \right|^2 + \sum_{1 \leq j < k \leq n} \alpha_j \alpha_k |x_j - x_k|^2 = \sum_{j=1}^n \alpha_j |x_j|^2. \quad (3.1)$$

**Theorem 3.1.** Let  $x_1, x_2, \dots, x_n$  be  $\tau$ -measurable operators and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be positive real numbers such that  $\sum_{j=1}^n \alpha_j = 1$  and  $f: [0, \infty) \rightarrow [0, \infty)$  such that  $f(0) = 0$  and  $g(t) = f(\sqrt{t})$  is convex on  $[0, \infty)$ . Then

$$f \left( \left| \sum_{j=1}^n \alpha_j x_j \right| \right) + \sum_{1 \leq j < k \leq n} f(\sqrt{\alpha_j \alpha_k} |x_j - x_k|) \preceq \sum_{j=1}^n \alpha_j f(|x_j|). \quad (3.2)$$

**Proof.** By (2.1), (2.3) and (3.1), we obtain that

$$\begin{aligned} \sum_{j=1}^n \alpha_j f(|x_j|) &= \sum_{j=1}^n \alpha_j g(|x_j|^2) \succeq g \left( \sum_{j=1}^n \alpha_j |x_j|^2 \right) \\ &= g \left( \left| \sum_{j=1}^n \alpha_j x_j \right|^2 + \sum_{1 \leq j < k \leq n} \alpha_j \alpha_k |x_j - x_k|^2 \right) \end{aligned}$$

$$\begin{aligned} & \preceq g\left(\left|\sum_{j=1}^n \alpha_j x_j\right|^2\right) + \sum_{1 \leq j < k \leq n} g\left(\alpha_j \alpha_k |x_j - x_k|^2\right) \\ & = f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right). \end{aligned} \quad \square$$

**Theorem 3.2.** Let  $x_1, x_2, \dots, x_n$  be  $\tau$ -measurable operators and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be positive real numbers such that  $\sum_{j=1}^n \alpha_j = 1$  and  $f: [0, \infty) \rightarrow [0, \infty)$  such that  $h(t) = f(\sqrt{t})$  is concave on  $[0, \infty)$ .

Then

$$\sum_{j=1}^n \alpha_j f(|x_j|) \preceq f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right). \quad (3.3)$$

**Proof.** We may assume  $f(0) = 0$ . Using (2.2), (2.4) and (3.1) we get that

$$\begin{aligned} \sum_{j=1}^n \alpha_j f(|x_j|) &= \sum_{j=1}^n \alpha_j h(|x_j|^2) \preceq h\left(\sum_{j=1}^n \alpha_j |x_j|^2\right) \\ &= h\left(\left|\sum_{j=1}^n \alpha_j x_j\right|^2 + \sum_{1 \leq j < k \leq n} \alpha_j \alpha_k |x_j - x_k|^2\right) \\ &\preceq h\left(\left|\sum_{j=1}^n \alpha_j x_j\right|^2\right) + \sum_{1 \leq j < k \leq n} h\left(\alpha_j \alpha_k |x_j - x_k|^2\right) \\ &= f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right). \end{aligned} \quad \square$$

Specializing Theorems 3.1 and 3.2 to the functions  $f(t) = t^p$  ( $2 \leq p < \infty$ ) and  $f(t) = t^p$  ( $0 \leq p \leq 2$ ).

**Corollary 3.1.** Let  $x_1, x_2, \dots, x_n$  be  $\tau$ -measurable operators and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be positive real numbers such that  $\sum_{j=1}^n \alpha_j = 1$ . Then

$$\left|\sum_{j=1}^n \alpha_j x_j\right|^p + \sum_{1 \leq j < k \leq n} (\alpha_j \alpha_k)^{\frac{p}{2}} |x_j - x_k|^p \preceq \sum_{j=1}^n \alpha_j |x_j|^p$$

for  $2 \leq p < \infty$ , and

$$\sum_{j=1}^n \alpha_j |x_j|^p \preceq \left|\sum_{j=1}^n \alpha_j x_j\right|^p + \sum_{1 \leq j < k \leq n} (\alpha_j \alpha_k)^{\frac{p}{2}} |x_j - x_k|^p$$



for  $0 \leq p \leq 2$ .

Applying Corollary 3.1 for the trace norm  $\|\cdot\|_1$ , we have the following result.

**Corollary 3.2.** Let  $x_1, x_2, \dots, x_n$  be  $\tau$ -measurable operators and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be positive real numbers such that  $\sum_{j=1}^n \alpha_j = 1$ . Then

$$\left\| \sum_{j=1}^n \alpha_j x_j \right\|_p^p + \sum_{1 \leq j < k \leq n} (\alpha_j \alpha_k)^{\frac{p}{2}} \|x_j - x_k\|_p^p \preceq \sum_{j=1}^n \alpha_j \|x_j\|_p^p$$

for  $2 \leq p < \infty$ , and

$$\sum_{j=1}^n \alpha_j \|x_j\|_p^p \preceq \left\| \sum_{j=1}^n \alpha_j x_j \right\|_p^p + \sum_{1 \leq j < k \leq n} (\alpha_j \alpha_k)^{\frac{p}{2}} \|x_j - x_k\|_p^p$$

for  $0 \leq p \leq 2$ . In particular

$$\left\| \sum_{j=1}^n x_j \right\|_p^p + \sum_{1 \leq j < k \leq n} \|x_j - x_k\|_p^p \preceq n^{p-1} \sum_{j=1}^n \|x_j\|_p^p$$

for  $2 \leq p < \infty$ , and

$$n^{p-1} \sum_{j=1}^n \|x_j\|_p^p \preceq \left\| \sum_{j=1}^n x_j \right\|_p^p + \sum_{1 \leq j < k \leq n} \|x_j - x_k\|_p^p$$

for  $0 \leq p \leq 2$ .

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#### **$\tau$ -ӨЛШЕМДІ ОПЕРАТОРЛАР ҮШІН КЛАРКСОННЫҢ ӘЛСІЗ МАЖОРЛАНҒАН ТЕҢСІЗДІКТЕРІ**

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**Тірек сөздер:** Кларксон теңсіздіктері,  $\tau$ -өлшемді оператор, Фон Нейман алгебрасы, субмажорланған.

**Андатпа.**  $(M, \tau)$  жартылай ақырлы фон Нейман алгебрасы және  $f - f(0) = 0$  болатын  $[0, \infty)$  теги теріс емес функция болсын.  $x_1, x_2, \dots, x_n$  дер  $\tau$  – өлшемді операторлар және  $\alpha_1, \alpha_2, \dots, \alpha_n$  оң нақты сандары  $\sum_{j=1}^n \alpha_j = 1$  болсын. Біз келесі нәтижелерді аламыз.

(1) Егер  $g(t) = f(\sqrt{t})$  дәнес болса, онда

$$f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right) \leq \sum_{j=1}^n \alpha_j f(|x_j|).$$

(2) Егер  $h(t) = f(\sqrt{t})$  ойыс болса, онда

$$\sum_{j=1}^n \alpha_j f(|x_j|) \leq f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right).$$

### НЕРАВЕНСТВА СЛАБО МАЖОРИЗАЦИОННЫЕ КЛАРКСОНА ДЛЯ $\tau$ -ИЗМЕРИМЫХ ОПЕРАТОРОВ

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**Ключевые слова:** неравенства Кларксона,  $\tau$ -измеримый оператор, алгебра Фон Неймана, субмажоризация.

**Аннотация.** Пусть  $(M, \tau)$  полу конечная алгебра Фон Неймана и  $f$  неотрицательная функция на  $[0, \infty)$  с  $f(0) = 0$ . Пусть  $x_1, x_2, \dots, x_n$   $\tau$  – измеримых операторов и  $\alpha_1, \alpha_2, \dots, \alpha_n$  положительные вещественные числа такие, что  $\sum_{j=1}^n \alpha_j = 1$ . Мы получили следующие результаты.

(1) Если  $g(t) = f(\sqrt{t})$  является выпуклым, то

$$f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right) \leq \sum_{j=1}^n \alpha_j f(|x_j|).$$

(2) Если  $h(t) = f(\sqrt{t})$  является вогнутой, то

$$\sum_{j=1}^n \alpha_j f(|x_j|) \leq f\left(\left|\sum_{j=1}^n \alpha_j x_j\right|\right) + \sum_{1 \leq j < k \leq n} f\left(\sqrt{\alpha_j \alpha_k} |x_j - x_k|\right).$$

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