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CLARKSON WEAK MAJORIZATION INEQUALITIES FOR τ-MEASURABLE OPERATORS

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Key words: Clarkson inequality, τ-measurable operator, von Neumann algebra, submajorization.

Abstract. Let (M,τ) be a semi-finite von Neumann algebra and f be a nonnegative function on $[0,\infty)$ with $f(0)=0$. Let $x_1, x_2, \cdots, x_n$ be τ-measurable operators and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^{n} \alpha_j = 1$. We have the following results.

(1) If $g(t)=f(\sqrt{t})$ is convex, then

$$f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right)+\sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right) \preceq \sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right).$$

(2) If $h(t)=f(\sqrt{t})$ is concave, then

$$\sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right) \preceq f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right)+\sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right).$$

1. Introduction. Let M_n be von Neumann algebra of $n \times n$ complex matrices, and let M_n^+ be positive part of M_n. Hirzallah and Kittaneh in [5] proved the following noncommutative Clarkson inequalities for $n-$tuples of operators: Let $|||\cdot|||$ be a unitarily norm, $A_0, \cdots, A_{n-1} \in \mathrm{M}_n^+$ and $\alpha_0, \cdots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j = 1$.

(1) If f be a nonnegative function on $[0, \infty)$ such that $f(0) = 0$ and $g(t) = f(\sqrt{t})$ is convex on $[0, \infty)$, then

$$\left|\left|\left| f\left(\left|\sum_{j=0}^{n-1} \alpha_j A_j\right|\right) + \sum_{0 \le j < k \le n-1} f\left(\sqrt{\alpha_j \alpha_k}\left|A_j - A_k\right|\right) \right|\right|\right| \le \left|\left|\left| \sum_{j=0}^{n-1} \alpha_j f\left(\left|A_j\right|\right) \right|\right|\right|.$$

(2) If f be a nonnegative function on $[0, \infty)$ such that $g(t) = f(\sqrt{t})$ is concave on $[0, \infty)$, then

$$\left|\left|\left| \sum_{j=0}^{n-1} \alpha_j f\left(\left|A_j\right|\right) \right|\right|\right| \le \left|\left|\left| f\left(\left|\sum_{j=0}^{n-1} \alpha_j A_j\right|\right) + \sum_{0 \le j < k \le n-1} f\left(\sqrt{\alpha_j \alpha_k}\left|A_j - A_k\right|\right) \right|\right|\right|.$$

The purpose of this paper is to extend the above results to $n-$tuples of $\tau-$measurable operators.

This paper is organized as follows. Section 2 contains some preliminary definitions.

In section 3, we proved the weak majorization type of Clarcson inequalities for n tuples of $\tau-$measurable operators.

2. Preliminaries. Throughout this paper, we denote by M a semi-finite von Neumann algebra on the Hilbert space H with a normalized normal faithful finite trace τ. The closed densely defined linear operator x in H with domain $D(x)$ is said to be affiliated with M if and only if $u^* x u = x$ for all unitary u which belong to the commutant M' of M. If x is affiliated with M, the x said to be $\tau-$measurable if for every $\varepsilon > 0$ there exists a projection $e \in M$ such that $e(H) \subseteq D(x)$ and $\tau(e^{\perp}) < \varepsilon$ (where for any projection e we let $e^{\perp} = 1 - e$). The set of all $\tau-$measure operators will be denoted by $L_0(M)$. The set $L_0(M)$ is a $*-$algebra with sum and product being the respective closure of the algebraic sum and product. Let $P(M)$ be the lattice of projections of M. The sets

$$\aleph(\varepsilon, \delta) = \left\{ x \in L_0(M) : \exists e \in P(M) \text{ such that } \|xe\| < \varepsilon,\ \tau(e^{\perp}) < \delta \right\}$$

$(\varepsilon, \delta > 0)$ from a base at 0 for an metrizable Hausdorff topology in $L_0(M)$ called the measure topology. Equipped with the measure topology, $L_0(M)$ is a complete topological $*-$algebra (see [6]). For $x \in L_0(M)$, the generalized singular value function $\mu(x)$ of x is defined by

$$\mu_s(x) = \inf\left\{ \|xe\| : e \in P(M),\ \tau(e^{\perp}) \le s \right\} \quad (s \ge 0).$$

If $x, y \in L_0(M)$, then we say that x is submajorized by y and write $x \preceq y$ if and only if

$$\int_0^t \mu_s(x) ds \le \int_0^t \mu_s(y) ds, \quad \forall t > 0.$$

As Proposition 4.6 in [3], we obtain the following result.

Lemma 2.1. Let f be a continuous increasing function on $\mathbf{R}_+$ with $f(0) = 0$. Let $x_1, x_2, \cdots, x_n$ be positive $\tau-$measurable operators and let $a_1, a_2, \cdots, a_n$ be positive elements in M with $\sum_{j=1}^{n} a_j^* a_j \le 1$.

(1) When f is convex, we have

$$\mu_s\left[f\left(\sum_{j=1}^{n}a_j^*x_ja_j\right)\right]\le\mu_s\left[\sum_{j=1}^{n}a_j^*f(x_j)a_j\right],\ s>0\ . \tag{2.1}$$

(2) When f is concave, we have

$$\mu_s\left[\sum_{j=1}^{n}a_j^*f(x_j)a_j\right]\le\mu_s\left[f\left(\sum_{j=1}^{n}a_j^*x_ja_j\right)\right],\ s>0\,. \tag{2.2}$$

The following lemma is a well-known result (see [2], Theorem 5.3.).

Lemma 2.2. Let $x_1,x_2,\cdots,x_n$ be positive $\tau-$measurable operators.

(1) If $g:[0,\infty)\to[0,\infty)$ be a convex function with $g(0)=0$. Then

$$\sum_{j=1}^{n}g(x_j)\preceq g\left(\sum_{j=1}^{n}x_j\right). \tag{2.3}$$

(2) If $h:[0,\infty)\to[0,\infty)$ be a concave function. Then

$$h\left(\sum_{j=1}^{n}x_j\right)\preceq\sum_{j=1}^{n}h(x_j). \tag{2.4}$$

3. Main result. Let $x_1,x_2,\cdots,x_n$ be positive $\tau-$measurable operators and let $\alpha_1,\alpha_2,\cdots,\alpha_n$ be positive real numbers such that $\sum_{j=1}^{n}\alpha_j=1$. Then

$$\left|\sum_{j=1}^{n}\alpha_jx_j\right|^2+\sum_{1\le j<k\le n}\alpha_j\alpha_k\left|x_j-x_k\right|^2=\sum_{j=1}^{n}\alpha_j\left|x_j\right|^2\ . \tag{3.1}$$

Theorem 3.1. Let $x_1,x_2,\cdots,x_n$ be $\tau-$measurable operators and let $\alpha_1,\alpha_2,\cdots,\alpha_n$ be positive real numbers such that $\sum_{j=1}^{n}\alpha_j=1$ and $f:[0,\infty)\to[0,\infty)$ such that $f(0)=0$ and $g(t)=f(\sqrt{t})$ is convex on $[0,\infty)$. Then

$$f\left(\left|\sum_{j=1}^{n}\alpha_jx_j\right|\right)+\sum_{1\le j<k\le n}f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right)\preceq\sum_{j=1}^{n}\alpha_jf\left(\left|x_j\right|\right). \tag{3.2}$$

Proof. By (2.1), (2.3) and (3.1), we obtain that

$$\sum_{j=1}^{n}\alpha_jf\left(\left|x_j\right|\right)=\sum_{j=1}^{n}\alpha_jg\left(\left|x_j\right|^2\right)\succeq g\left(\sum_{j=1}^{n}\alpha_j\left|x_j\right|^2\right)$$

$$=g\left(\left|\sum_{j=1}^{n}\alpha_jx_j\right|^2+\sum_{1\le j<k\le n}\alpha_j\alpha_k\left|x_j-x_k\right|^2\right)$$

$$\succeq g\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|^2\right)+\sum_{1\le j<k\le n} g\left(\alpha_j\alpha_k\left|x_j-x_k\right|^2\right)$$

$$=f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right)+\sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right). \qquad \square$$

Theorem 3.2. Let $x_1, x_2, \cdots, x_n$ be $\tau-$measurable operators and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^{n}\alpha_j = 1$ and $f:[0,\infty)\to[0,\infty)$ such that $h(t)=f(\sqrt{t})$ is concave on $[0,\infty)$. Then

$$\sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right)\preceq f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right)+\sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right). \qquad (3.3)$$

Proof. We my assume $f(0)=0$. Using (2.2), (2.4) and (3.1) we get that

$$\sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right)=\sum_{j=1}^{n}\alpha_j h\left(\left|x_j\right|^2\right)\preceq h\left(\sum_{j=1}^{n}\alpha_j\left|x_j\right|^2\right)$$

$$=h\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|^2+\sum_{1\le j<k\le n}\alpha_j\alpha_k\left|x_j-x_k\right|^2\right)$$

$$\preceq h\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|^2\right)+\sum_{1\le j<k\le n} h\left(\alpha_j\alpha_k\left|x_j-x_k\right|^2\right)$$

$$=f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right)+\sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right) \qquad .\square$$

Specializing Theorems 3.1 and 3.2 to the functions $f(t)=t^p \ (2\le p<\infty)$ and $f(t)=t^p \ (0\le p\le 2)$.

Corollary 3.1. Let $x_1, x_2, \cdots, x_n$ be $\tau-$measurable operators and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^{n}\alpha_j = 1$. Then

$$\left|\sum_{j=1}^{n}\alpha_j x_j\right|^p+\sum_{1\le j<k\le n}\left(\alpha_j\alpha_k\right)^{\frac{p}{2}}\left|x_j-x_k\right|^p\preceq\sum_{j=1}^{n}\alpha_j\left|x_j\right|^p$$

for $2\le p<\infty$, and

$$\sum_{j=1}^{n}\alpha_j\left|x_j\right|^p\preceq\left|\sum_{j=1}^{n}\alpha_j x_j\right|^p+\sum_{1\le j<k\le n}\left(\alpha_j\alpha_k\right)^{\frac{p}{2}}\left|x_j-x_k\right|^p$$

for $0 \le p \le 2$.

Applying Corollary 3.1 for the trace norm $\|\cdot\|_1$, we have the following result.

Corollary 3.2. Let $x_1, x_2, \cdots, x_n$ be $\tau-$measurable operators and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^{n} \alpha_j = 1$. Then

$$\left\|\sum_{j=1}^{n} \alpha_j x_j\right\|_p^p + \sum_{1 \le j < k \le n} (\alpha_j \alpha_k)^{\frac{p}{2}} \|x_j - x_k\|_p^p \preceq \sum_{j=1}^{n} \alpha_j \|x_j\|_p^p$$

for $2 \le p < \infty$, and

$$\sum_{j=1}^{n} \alpha_j \|x_j\|_p^p \preceq \left\|\sum_{j=1}^{n} \alpha_j x_j\right\|_p^p + \sum_{1 \le j < k \le n} (\alpha_j \alpha_k)^{\frac{p}{2}} \|x_j - x_k\|_p^p$$

for $0 \le p \le 2$. In particular

$$\left\|\sum_{j=1}^{n} x_j\right\|_p^p + \sum_{1 \le j < k \le n} \|x_j - x_k\|_p^p \preceq n^{p-1} \sum_{j=1}^{n} \|x_j\|_p^p$$

for $2 \le p < \infty$, and

$$n^{p-1} \sum_{j=1}^{n} \|x_j\|_p^p \preceq \left\|\sum_{j=1}^{n} x_j\right\|_p^p + \sum_{1 \le j < k \le n} \|x_j - x_k\|_p^p$$

for $0 \le p \le 2$.

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τ-ӨЛШЕМДІ ОПЕРАТОРЛАР ҮШІН КЛАРКСОННЫҢ ӘЛСІЗ МАЖОРЛАНҒАН ТЕҢСІЗДІКТЕРІ

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Тірек сөздер: Кларксон теңсіздіктері, *τ*-өлшемді оператор, Фон Нейман алгебрасы, субмажорланған.

Андатпа. (M,τ) жартылай ақырлы фон Нейман алгебрасы және f _ $f(0)=0$ болатын $[0,\infty)$ тегі теріс емес функция болсын. $x_1,x_2,\cdots,x_n$ дер $\tau-$ өлшемді операторлар және $\alpha_1,\alpha_2,\cdots,\alpha_n$ оң нақты сандары $\sum_{j=1}^{n}\alpha_j=1$ болсын . Біз келесі нәтижелерді аламыз.

(1) Егер $g(t)=f(\sqrt{t})$ дөңес болса, онда

$$f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right)+\sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right)\preceq\sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right).$$

(2) Егер $h(t)=f(\sqrt{t})$ ойыс болса, онда

$$\sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right)\preceq f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right)+\sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right).$$

НЕРАВЕНСТВА СЛАБО МАЖОРИЗАЦИОННЫЕ КЛАРКСОНА ДЛЯ τ-ИЗМЕРИМЫХ ОПЕРАТОРОВ

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Ключевые слова: неравенства Кларксона, τ-измеримый оператор, алгебра Фон Неймана, субмажоризация.

Аннотация. Пусть (M,τ) полу конечная алгебра Фон Неймана и f неотрицательная функция на $[0,\infty)$ с $f(0)=0$. Пусть $x_1,x_2,\cdots,x_n$ $\tau-$ измеримых операторов и $\alpha_1,\alpha_2,\cdots,\alpha_n$ положительные вещественные числа такие, что $\sum_{j=1}^{n}\alpha_j=1$. Мы получили следушие резултаты.

(1) Если $g(t)=f(\sqrt{t})$ является выпуклым, то

$$f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right)+\sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right)\preceq\sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right).$$

(2) Если $h(t)=f(\sqrt{t})$ является вогнутой, то

$$\sum_{j=1}^{n}\alpha_j f\left(\left|x_j\right|\right)\preceq f\left(\left|\sum_{j=1}^{n}\alpha_j x_j\right|\right)+\sum_{1\le j<k\le n} f\left(\sqrt{\alpha_j\alpha_k}\left|x_j-x_k\right|\right).$$

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