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# A NONLOCAL BOUNDARY VALUE PROBLEM ON THE HEISENBERG GROUP

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**Keywords:** nonlocal boundary value problem, Kohn Laplacian, Heisenberg group.

**Abstract.** In this note we construct a well-posed boundary value problem for the Kohn Laplacian with nonlocal boundary conditions in a bounded smooth domain on the Heisenberg group. It is also showed that the boundary value problem is explicitly solvable for any bounded smooth domain.

**1. Introduction.** In a bounded domain of the Euclidean space $\Omega \in \mathbb{R}^d, d \geq 2$, it is very well known that the solution to the Laplacian equation

$$\Delta u(x) = f(x), \ x \in \Omega, \tag{1}$$

is given by the Green formula (or the Newton potential formula)

$$u(x) = \int_\Omega \varepsilon_d(x-y) f(y) dy, \quad x \in \Omega, \tag{2}$$

for suitable functions $f$ supported in Ω. Here $\varepsilon_d$ is the fundamental solution to $\Delta$ in $\mathbb{R}^d$ given by

$$\varepsilon_d(x-y) = \begin{cases} \frac{1}{(2-d)s_d} \frac{1}{|x-y|^{d-2}}, & d \geq 3, \\ \frac{1}{2\pi} \log|x-y|, & d = 2, \end{cases} \tag{3}$$

where $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ is the surface area of the unit sphere in $\mathbb{R}^d$.

An interesting question having several important applications is what boundary conditions can be put on $u$ on the (smooth) boundary $\partial\Omega$ so that equation (1) complemented by this boundary condition would have the solution in Ω still given by the same formula (2), with the same kernel $\varepsilon_d$ given by (3). It turns out that the answer to this question is the integral boundary condition

$$-\frac{1}{2}u(x) + \int_{\partial\Omega} \frac{\partial \varepsilon_d(x-y)}{\partial n_y} u(y) dS_y - \int_{\partial\Omega} \varepsilon_d(x-y) \frac{\partial u(y)}{\partial n_y} dS_y = 0, x \in \partial\Omega, \tag{4}$$

where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at a point $y$ on $\partial\Omega$. A converse question to the one above would be to determine the trace of the Newton potential (2) on the boundary surface $\partial\Omega$, and one can use the potential theory to show that it has to be given by (4).

The boundary condition (4) appeared in M. Kac's work [10] where he called it and the subsequent spectral analysis the principle of not feeling the boundary". This was further expanded in Kac's book [11] with several further applications to the spectral theory and the asymptotics of the Weyl's eigenvalue counting function. In [12] by using the boundary condition (1.4) the eigenvalues and eigenfunctions of the

Newton potential (2) were explicitly calculated in the 2-disk and in the 3-ball. In general, the boundary value problem (1)-(4) has various interesting properties and applications (see, for example, Kac [10, 11] and Saito [19]). The boundary value problem (1)-(4) can also be generalized for higher degrees of the Laplacian, see [13, 14].

In this note we are interested in and give answers to the following questions:

– What happens if an elliptic operator (the Laplacian) in (1) is replaced by a hypoelliptic operator? We will realize this as a model of replacing the Euclidean space by the Heisenberg group and the Laplacian on $\mathbb{R}^d$ by a sub-Laplacian (or the Kohn-Laplacian) on $\mathbb{H}_{n-1}$. We will show that the boundary condition (4) is replaced by the integral boundary condition (15) in this setting (see also (11)).

– Since the theory of boundary value problems for elliptic operators is well understood, we know that the single condition (4) on the boundary $\partial\Omega$ of a bounded domain $\Omega$ guarantees the unique solvability of the equation (1) in $\Omega$. Is this uniqueness preserved in the hypoelliptic model as well for a suitably chosen replacement of the boundary condition (4)? The case of the second order operators is favourable from this point of view due to the validity of the maximum principle, see Bony [1]. The Dirichlet problem has been considered by Jerison [9]. The answer in the case of the boundary value problem in our setting is given in Theorem 2.1.

We now describe the setting of this paper. The Heisenberg group $\mathbb{H}_{n-1}$ is the space $\mathbb{C}^{n-1}\times\mathbb{R}$ with the group operation given by

$$(\zeta,t)\circ(\eta,\tau)=(\zeta+\eta,t+\tau+2\mathrm{Im}\,\zeta\eta), \tag{5}$$

for f$(\zeta,t),(\eta,\tau)\in\mathbb{C}^{n-1}\times\mathbb{R}$. Writing $\zeta=x+iy$, with $x_j,y_j,j=1,\dots,n-1$, the real coordinates on $\mathbb{H}_{n-1}$, the left-invariant vector fields

$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},j=1,\dots,n-1,$$

$$Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t},j=1,\dots,n-1,$$

$$T=\frac{\partial}{\partial \tau}$$

form a basis for the Lie algebra $\eta_{n-1}$ $of$ $\mathbb{H}_{n-1}$.

On the other hand, $\mathbb{H}_{n-1}$ can be viewed as the boundary of the Siegel upper half space in $\mathbb{C}^n$, $\mathbb{H}_{n-1}=\{(\zeta,z_n)\in\mathbb{C}^n:\mathrm{Im}\,z_n=|\zeta|^2,\ \zeta=(z_1,\dots,z_{n-1})\}$. Parameterizing $\mathbb{H}_{n-1}$ by $z=(\zeta,t)$, where $\mathbb{H}_{n-1}$ $t=\mathrm{Re}\,z_n$, a basis for the complex tangent space of $\mathbb{H}_{n-1}$ at the point $z$ is given by the left-invariant vector fields

$$X_j=\frac{\partial}{\partial z_j}+i\bar{z}\frac{\partial}{\partial t},j=1,\dots,n-1.$$

We denote their conjugates by $X_{\bar{j}}=\bar{X}_j=\frac{\partial}{\partial \bar{z}_j}-iz\frac{\partial}{\partial t}$. The operator

$$\square_{a,b}=\sum_{j=1}^{n-1}\left(aX_jX_{\bar{j}}+bX_{\bar{j}}X_j\right),a+b=n-1, \tag{6}$$

is a left-invariant, rotation invariant differential operator that is homogeneous of degree two (cf. [3]). This operator is a slight generalisation of the standard sub-Laplacian or Kohn-Laplacian $\square_b$ on the Heisenberg group $\mathbb{H}_{n-1}$ which, when acting on the coefficients of a (0; q)-form can be written as

$$\square_b=-\frac{1}{n-1}\sum_{j=1}^{n-1}\left((n-1-q)\,X_jX_{\bar{j}}+qX_{\bar{j}}X_j\right).$$

Folland and Stein [4] found that a fundamental solution of the operator $\Box_{a,b}$ is a constant multiple of

$$\varepsilon(z) = \varepsilon(\zeta, t) = \frac{1}{(t+i|\zeta|^2)^a (t-i|\zeta|^2)^b}, \tag{7}$$

and defined the Newton potential (volume potential) for a function *f* with compact support contained in a set $\Omega \in \mathbb{H}_{n-1}$ by

$$u(z) = \int_{\Omega} f(\xi)\varepsilon(\xi^{-1}z)dv(\xi), \tag{8}$$

with dv being the volume element (the Haar measure on $\mathbb{H}_{n-1}$), coinciding with the Lebesgue measure on $\mathbb{C}^{n-1} \times \mathbb{R}$. More precisely, they proved that

$$\Box_{a,b} u = c_{a,b} f,$$

where the constant $c_{a,b}$ is zero if a and b= - 1, - 2,..., n, n+1,..., and $c_{a,b} \neq 0$ if $\alpha$ or b$\neq$ -1,-2,..., n, n+1,... In fact, then we can take

$$c_{a,b} = \frac{2(a^2+b^2)\mathrm{Vol}(B_1)}{(2i)^n} \frac{(n-1)!}{a(a-1)\dots(a-n+1)} (1-\exp(-2ia\pi))$$

for a$\notin$ Z, see the proof of Theorem 1.6 in Romero [15]. Similar conclusions by a different methods were obtained by Greiner and Stein [8]. For a more general analysis of fundamental solutions for sub-Laplacian we can refer to Folland [5] as well as to a discussion and references in Stein [20].

In the above notation, the distribution $\frac{1}{c_{a,b}}\varepsilon$ - is the fundamental solution of $\Box_{a,b}$, while $\varepsilon$ satisfies the equation

$$\Box_{a,b}\, \varepsilon = c_{a,b}\, \delta. \tag{9}$$

However, although we could have rescaled $\varepsilon$ for it to become the fundamental solution, we prefer to keep the notation yielding (9) in order to follow the notation of [4] and [15] to be able to refer to their results directly.

In this paper we assume that $c_{a,b} \neq 0$,, i.e. both α and $b \neq -1,-2,\dots,n,n+1,\dots$. In addition, without loss of generality we may also assume that $a, b \geq 0$.

Now, in analogy to the elliptic boundary value problem (1)-(4) for the Laplacian Δ in $\mathbb{R}^d$, we consider the hypoelliptic boundary value problem for the sub-Laplacian $\Box_{a,b}$ on $\mathbb{H}_{n-1}$, namely the equation

$$\Box_{a,b} u = c_{a,b} f \tag{10}$$

in a bounded set $\Omega \subset \mathbb{H}_{n-1}$ with smooth boundary $\partial\Omega$. The aim of this paper is to find a boundary condition of the Newton potential u on $\partial\Omega$, such that with this boundary condition the equation (10) has a unique solution, which is the Newton potential (8).

Basing our arguments on the analysis of Folland and Stein [4] and Romero [15] we show that the boundary condition (4) for the Laplacian in $\mathbb{R}^d$ is now replaced by the integral boundary condition (5) in this setting, namely by the condition

$$(c_{a,b} - H.R(z))u(z) - \int_{\partial\Omega} \varepsilon(\xi, z)\langle \nabla^{b,a} u(\xi), dv(\xi)\rangle + p.v.Wu(z) = 0, z \in \partial\Omega,$$

on the boundary $\partial\Omega$, where H.R (z) is the so-called half residue, and where the second and the third term can be interpreted as coming from the suitably defined respectively single and double layer potentials S and Wfor the problem. See Section 2 for the definitions and the precise statement. In Section 2 by using properties of fundamental solutions we construct a well-posed boundary value problem for the differential equation (10) with the required properties.

**2. The Kohn Laplacian.** Let $\Omega \subset \mathbb{H}_{n-1}$ be an open bounded domain with a smooth boundary $\partial\Omega \in C^{\infty}$. Consider the following analogy of the Newton potential on the Heisenberg group

$$u(z) = \int_{\Omega} f(\xi)\varepsilon(\xi, z)dv(\xi) \text{ в } \Omega, \tag{11}$$

where $\varepsilon(\xi,z)=\varepsilon(\xi^{-1}z)$ is the rescaled fundamental solution (1.7) of the sub-Laplacian, satisfying (1.9). As we mentioned u is a solution of (10) in $\Omega$. The aim of this section is to find a boundary condition for u such that with this boundary condition the equation (10) has a unique solution in $C^2(\Omega)$, say, and this solution is the Newton potential (11).

We recall a few notions and properties first. For z = (ζ,t) $\in \mathbb{H}_{n-1}$, we define its norm by $|z|:=(|\zeta|^4+|t|^2)^{\frac{1}{4}}$. As any (quasi-)norm on $\mathbb{H}_{n-1}$, this satisfies a triangle inequality with a constant, and allows for a polar decomposition. For 0 <a<1, Folland and Stein [4] defined the anisotropic Holder spaces $\Gamma_\alpha\{\Omega\}$ by

$$\Gamma_\alpha\{\Omega\}=\left\{f:\Omega\to\mathbb{C}:\ \sup_{\substack{z_1,z_2\in\Omega\\ z_1\neq z_2}}\frac{|f(z_2)-f(z_1)|}{|z_2^{-1}z_1|^{\alpha}}<\infty\right\}.$$

For $k\in\mathbb{N}$ and 0 < a < 1, one defines $\Gamma_{k+\alpha}\{\Omega\}$ as the space of all f: $\Omega\to\mathbb{C}$ such that all complex derivatives of f of order $k$ belong to $\Gamma_\alpha\{\Omega\}$.

A starting point for us will be that if $f\in\Gamma_\alpha(\Omega)$ for $\alpha<0$ then u defined by (11) is twice differentiable in the complex directions and satisfies the equation $\square_{a,b}u=c_{a,b}f$. We refer to Folland and Stein [4], Greiner and Stein [8], and to Romero [15] for three different approaches to this property. Moreover, Folland and Stein have shown that if $f\in\Gamma_\alpha(\Omega,\mathrm{loc})$ and $\square_{a,b}u=c_{a,b}f$, then $f\in\Gamma_{\alpha+2}(\Omega,\mathrm{loc})$. These results extend those known for the Laplacian, in suitably rede_ned anisotropic Holder spaces.

We record relevant single and double layer potentials for the problem (10). In [9], Jerison used the single layer potential defined by

$$S_0g(z)=\int_{\partial\Omega}g(\xi)\varepsilon(\xi,z),dS(\xi),$$

which, however, is not integrable over characteristic points. On the contrary, the functional

$$Sg(z)=\int_{\partial\Omega}g(\xi)\varepsilon(\xi,z)\langle X_j,dv(\xi)\rangle,$$

where $\langle X_j,dv\rangle$ is the canonical pairing between vector fields and differential forms, is integrable over the whole boundary $\partial\Omega$. Moreover, it was shown in [16, Theorem 2.3] that if the density of g(ξ) $\langle X_j,dv\rangle$ in the operator S is bounded then $Sg\in\Gamma_\alpha(\mathbb{H}_{n-1})$ for all $\alpha<1$. Parallel to S, it is natural to use the operator

$$Wu(z)=\int_{\partial\Omega}u(\xi)\langle\nabla^{b,a}\varepsilon(\xi,z),dv(\xi)\rangle \qquad (12)$$

as a double layer potential. Our main result:

Theorem 2.1. Let $\varepsilon(\xi,z)=\varepsilon(\xi^{-1}z)$ be the rescaled fundamental solution to $\square_{a,b}$, so that

$$\square_{a,b}\varepsilon=c_{a,b}\,\delta \quad on\ \mathbb{H}_{n-1} \qquad (13)$$

for any $f\in\Gamma_\alpha(\Omega)$, the Newton potential (11) is the unique solution in $C^2(\Omega)\cap C^1(\bar\Omega)$ of the equation

$$\square_{a,b}u=c_{a,b}f \qquad (14)$$

with the boundary condition

$$(c_{a,b}-H.R(z))u(z)+\lim_{\delta\to0}\int_{\partial\Omega\setminus\{|\xi^{-1}z|<\delta\}}u(\xi)\nabla^{a,b}\varepsilon(\xi,z),dv(\xi)) \qquad (15)$$

$$-\int_{\partial\Omega}\varepsilon(\xi,z),\langle\nabla^{a,b}\varepsilon(\xi,z),dv(\xi)\rangle=0, \qquad \text{для } z\in\partial\Omega,$$

where H:R(z) is the so-called half residue given by the formula

$$\mathrm{H.R}(z)=\lim_{\delta\to0}\int_{\partial\Omega\setminus\{|\xi^{-1}z|<\delta\}}\langle\nabla^{a,b}\varepsilon(\xi,z),dv(\xi)\rangle, \qquad (16)$$

c

$$\nabla^{a,b} g = \sum_{j-1}^{n-1} (aX_j g X_{\bar{j}} + bX_{\bar{j}} g X_j).$$

The half residue H.R.(z) in (16) appears in the jump relations for the problem (14) in the following way. The double layer potential Wu in (12) has two limits

$$W^{+}u(z) = \lim_{\substack{z_0 \to z \\ z_0 \in \Omega}} \int_{\partial\Omega} u(\xi)\langle \nabla^{a,b} \varepsilon(\xi, z_0), dv(\xi)\rangle$$

and

$$W^{-}u(z) = \lim_{\substack{z_0 \to z \\ z_0 \notin \Omega}} \int_{\partial\Omega} u(\xi)\langle \nabla^{a,b} \varepsilon(\xi, z_0), dv(\xi)\rangle,$$

and the principal value

$$W^{0}u(z) = p.v.\, Wu(z) = \lim_{\delta\to 0} \int_{\partial\Omega \setminus \{|\xi^{-1}z<\delta\}} u(\xi)\langle \nabla^{ab} \varepsilon(\xi, z), dv(\xi)\rangle.$$

We note that this principal value enters as the second term in the integral boundary condition (15). It was proved in [16, Theorem 2.4] that for sufficiently regular u (e.g. u $\in \Gamma_\alpha(\Omega)$) and z $\in \partial\Omega$ these limits exist and satisfy the jump relations

$$W^{+}u(z) - W^{-}u(z) = c_{a,b}\, u(z),$$

$$W^{0}u(z) - W^{-}u(z) = H.R.(z)u(z),$$

$$W^{+}u(z) - W^{0}u(z) = \left(c_{a,b} - H.R.(z)\right)u(z), \quad (17)$$

the last property (17) following from the first two by subtraction.

Proof of Theorem 2.1. Since the solid potential

$$u(z) = \int_{\partial\Omega} u(\xi)\langle \nabla^{b,a} \varepsilon(\xi, z), dv(\xi)\rangle \quad (18)$$

is a solution of (14), from the aforementioned results of Folland and Stein it follows that u is locally in $\Gamma_{\alpha+2}(\Omega, loc)$ and that it is twice complex differentiable in Ω. In particular, it follows that u $\in C^2(\Omega) \cap C^1(\bar{\Omega})$.

The following representation formula can be derived from the generalised second Green's formula (see Theorem 4.5 in [15] and cf. [15]), for $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ we have

$$c_{a,b}\, u(z) = c_{a,b} \int_{\Omega} f(\xi)\langle \varepsilon(\xi, z), dv(\xi)\rangle + \int_{\partial\Omega} u(\xi)\langle \nabla^{b,a} \varepsilon(\xi, z), dv(\xi)\rangle \quad (19)$$

$$- \int_{\partial\Omega} \varepsilon(\xi)\langle \nabla^{b,a} u(\xi, z), dv(\xi)\rangle \quad \text{for any } z \in \Omega.$$

Since $u(z)$ given by (18) is a solution of (14), using it in (19) we get

$$\int_{\partial\Omega} u(\xi)\langle \nabla^{a,b} \varepsilon(\xi, z_0), dv(\xi)\rangle - \int_{\partial\Omega} \varepsilon(\xi, z)\langle \nabla^{ab,a} u(\xi) dv(\xi)\rangle = 0 \quad (20)$$

for any $z \in \Omega$

It is easy to see that the fundamental solution, i.e. the function $\varepsilon(z)$ in (7) is homogeneous of degree -2n+2, that is

$$\varepsilon(\lambda, z) = \lambda^{-2a-2b}\varepsilon(z) = \lambda^{-2n+2}\varepsilon(z) \text{ for any } \lambda > 0,$$

since $a + b = n - 1$. It follows that and its first order complex derivatives are locally integrable. Since $\varepsilon(\xi, z) = \varepsilon(\xi^{-1}, z)$, we obtain that as z approaches the boundary, we can pass to the limit in the second term in (20).

By using this and the relation (17) as $z \in \Omega$ approaches the boundary $\partial\Omega$ from inside, we find that

$$(c_{a,b} - H.R(z))u(z) + \lim_{\delta\to 0} \int_{\partial\Omega \setminus \{|\xi^{-1}z\}} u(\xi)\langle \nabla^{a,b} \varepsilon(\xi, z), dv(\xi)\rangle \quad (21)$$

$$- \int_{\partial\Omega} \varepsilon(\xi, z)\langle \nabla^{b,a} u(\xi) dv(\xi)\rangle = 0, \text{ for any } z \in \partial\Omega.$$

This shows that (11) is a solution of the boundary value problem (14) with the boundary condition (15).

Now let us prove its uniqueness. If the boundary value problem has two solutions u and $u_1$, then the function $w = u - u_1 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies the homogeneous equation

$$\square_{a,b} w = 0 \text{ in } \Omega, \tag{22}$$

and the boundary condition (15), i.e.

$$c_{a,b} - H.R(z))w(z) + \lim_{\delta\to 0} \int_{\partial\Omega\setminus\{|\xi - z| < \delta\}} w(\xi)\langle\nabla^{a,b}\varepsilon(\xi,z), dv(\xi)\rangle - \int_{\partial\Omega}\varepsilon(\xi,z)\langle\nabla^{a,b}w(\xi)dv(\xi)\rangle = 0 \tag{23}$$

for any $z \in \partial\Omega$.

Since $f \equiv 0$ in this case instead of (19) we have the following representation formula

$$c_{a,b}\, w(z) = \int_{\partial\Omega} w(\xi,z)\langle\nabla^{a,b}\varepsilon(\xi)dv(\xi)\rangle - \int_{\partial\Omega}\varepsilon(\xi,z)\langle\nabla^{b,a}w(\xi)dv(\xi)\rangle \tag{24}$$

for any $z \in \partial\Omega$. As above, by using the properties of the double and single layer potentials as $z \to \partial\Omega$, we obtain

$$c_{a,b}\, w(z) = c_{a,b} - H.R(z))w(z) + \lim_{\delta\to 0} \int_{\partial\Omega\setminus\{|\xi - z| < \delta\}} w(\xi)\langle\nabla^{a,b}\varepsilon(\xi,z), dv(\xi)\rangle - \int_{\partial\Omega}\varepsilon(\xi,z)\langle\nabla^{a,b}w(\xi)dv(\xi)\rangle \tag{25}$$

for any $z \in \partial\Omega$. Comparing this with (23) we arrive at

$$w(z) = 0, z \in \partial\Omega. \tag{26}$$

The homogeneous equation (22) with the Dirichlet boundary condition (26) has only trivial solution $w \equiv 0$ in $\Omega$. This shows that the boundary value problem (14) with the boundary condition (15) has a unique solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$. This completes the proof of Theorem 2.1.

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## ГЕЙЗЕНБЕРГ ГРУППАСЫНДАҒЫ ЛОКАЛДЫ ЕМЕС ШЕКАРАЛЫҚ ЕСЕП

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**Тірек сөздер:** локалды емес шекаралық есеп, Кон Лапласы, Гейзенберг группасы.

**Аннотация.** Жұмыста Гейзенберг группасындағы шеңелген тегіс облыста локалды емес шекаралық шартты Кон Лаплас теңдеуіне қисында шекралық есебі құрастырылды. Және де кез келген шеңелген тегіс облыста алынған шекаралық есеп айқын түрде шешілетіндігі көрсетілді.

## ОБ ОДНОЙ НЕЛОКАЛЬНОЙ КРАЕВОЙ ЗАДАЧЕ НА ГРУППЕ ГЕЙЗЕНБЕРГА

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**Ключевые слова:** нелокальная, краевая задача, Кон Лапласа, группа Гейзенберга.

**Аннотация.** В статье раскрывается построение корректной краевой задачи для оператора Кон-Лапласа с нелокальным краевым условием в ограниченной гладкой области на группе Гейзенберга. А также показано, что краевая задача решаема в явном виде для любой ограниченной гладкой области.

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