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ASYMPTOTICAL REPRESENTATION OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATIONS

Annotation. We consider the two-point boundary value problem for the singularly perturbed higher order linear integro-differential equation. An asymptotic in a small parameter representation of the solution is obtained. It is shown that the solution has the phenomenon of the m-th order initial jump at the point $t = 0$.

Keywords: singularly perturbation, integro-differential equations, small parameter, initial jump.

1. Introduction

Applied mathematics studies natural processes using mathematical models for them. Any mathematical model is somewhat approximate. It is not absolutely adequate for the process it describes. Deriving the mathematical model, one tries to capture all essential, dominant feature of the process. On the other hand, the model should be “simple” enough to allow the analytical and/or numerical treatment leading to the information one wants to obtain about the process. A variety of models in physics, chemical kinetics, mathematical biology, and many other fields are quite naturally formulated in terms of differential equations. During the derivation of the model equation, some terms whose influence on the process is supposed to be negligible are often not taken into account. As a result, the model might be simplified considerably. Such simplifications often rely on physical intuition.

Perturbations that occur in different problem can be formally divided into two classes: *regular perturbations* and *singular perturbations.* Before giving formal definitions, we mention that the primary qualitative difference between these two kinds of perturbations is that regular perturbations lead to small changes from the solution of the unperturbed problem. Unlike such regular perturbations, singular perturbations, thought to be small sense, cause considerable change in the solutions.

Theory of asymptotic integration of singularly perturbed equations has become purposefully developed starting with the works of L. Schlesinger [1], G. D. Birkhoff [2], P. Noaillon [3]. In a further development of the main trends of the theory W. Wasow [4], A. N. Tikhonov [5], M. I. Vishik, L. A. Lusternik [6, 7], R. O’Malley [8], A. B. Vasil’eva, V. F. Butuzov [9], K. A. Kassymov [10, 11] and others have made a significant contribution. In the works of M. I. Vishik, L. A. Lyusternik [6] and K. A. Kassymov [11] first studied initial problems for singularly perturbed nonlinear equations of the second order with unbounded initial conditions when the small parameter tends to zero. These problems are called the Cauchy problems with an initial jump. In these problems, the solutions of singularly perturbed equations tend to the solutions of the degenerate equations with the modified initial conditions. The Cauchy problems with initial jumps for integro-differential equations was considered in [12, 13].

The present work is devoted to research asymptotic in the small parameter the behavior of solutions of the unseparated two-point boundary value problem for a singularly perturbed linear integro-differential equations of n-th order with integral terms of Fredholm type, we have that the solution of the boundary value problem has m-th order initial jump at point t=0. Boundary problems for singularly perturbed integro-differential equations of higher order with the initial jump of the m_1 −th order was considered in [14].

2. Statement of the problem Preliminary materials

Consider the following singularly perturbed integro-differential equation

$$L_\varepsilon y \equiv \varepsilon y^{(n)} + A_1(t)y^{(n-1)} + ... + A_n(t)y = F(t) + \int_0^1 \sum_{i=0}^{m+1} H_i(t,x)y^{(i)}(x,\varepsilon)dx, \quad (1)$$

with boundary conditions

$$h_i y(t,\varepsilon) \equiv \sum_{j=0}^{m} \alpha_{ij} y^{(j)}(0,\varepsilon) + \sum_{j=0}^{r} \beta_{ij} y^{(j)}(1,\varepsilon) = a_i, \quad i = \overline{1,n}, \quad m < n-1, \ r < n-1, \quad (2)$$

where $\varepsilon > 0$ is a small parameter, α_{ij}, β_{ij}, $a_i \in R$ are known constants, independent of ε.

Assume that the following conditions hold:

I. Functions $A_i(t), F(t), i = \overline{1,n}$, are sufficiently smooth and defined on the interval [0,1];

II. $A_1(t) \ge \gamma = const > 0, \ 0 \le t \le 1$;

III. Functions $H_i(t,x), i = \overline{0, m+1}$ are defined in the domain $D = (0 \le t \le 1, 0 \le x \le 1)$ and sufficiently smooth;

IV. $$\bar{\Delta} = \begin{vmatrix} h_1 y_{10}(t) & ... & h_1 y_{n-1,0}(t) & \alpha_{1m} \\ ... & ... & ... & ... & ... \\ h_n y_{10}(t) & ... & h_n y_{n-1,0}(t) & \alpha_{nm} \end{vmatrix} \ne 0 .$$

For the fundamental system of solutions of singularly perturbed homogeneous differential equation

$$L_\varepsilon y \equiv \varepsilon\, y^{(n)} + A_1(t)y^{(n-1)} + ... + A_n(t)y = 0 \quad (3)$$

the following asymptotic representation holds as $\varepsilon \to 0$:

$$\begin{cases} y_i^{(j)}(t,\varepsilon) = y_{i0}^{(j)}(t) + O(\varepsilon), \quad i = 1,...,n-1, \ j = 0,...,n-1, \\ y_n^{(j)}(t,\varepsilon) = \dfrac{1}{\varepsilon^j} \exp\left(\dfrac{1}{\varepsilon}\int_0^t \mu(x)dx\right) \cdot (\mu^j(t) y_{n0}(t) + O(\varepsilon)), \ j = 0,...,n-1, \end{cases} \quad (4)$$

where $\mu(t) = -A_1(t) < 0$, functions $y_{i0}(t), i = 0,...,n-1$ are solutions of the problem:

$$L_0 y_{io} \equiv A_1(t) y_{i0}^{(n-1)}(t) + ... + A_n(t) y_{i0}(t) = 0, \ y_{i0}^{(j)}(0) = \delta_{i-1,\,j}, \ i = 1,...,n-1, j = 0,...,n-2,$$

and

$$y_{n0}(t) = \left(A_1(0)/A_1(t)\right)^{n-1} \exp\left(\int_0^t \left(A_2(x)/A_1(x)\right)dx\right). \quad (5)$$

The proof of the formulas (4) are readily obtained from the known theorems of L. Schlesinger [1], D. Birkhoff [2] and G P. Noaillon [3].

Let the function $K(t,s,\varepsilon)$, $0 \le s \le t \le 1$, be a solution of the following problem:

$$L_\varepsilon K(t,s,\varepsilon) = 0,\ K^{(j)}(s,s,\varepsilon) = 0, \quad j = 0,...,n-2, \quad K^{(n-1)}(s,s,\varepsilon) = 1.$$

Function $K(t,s,\varepsilon)$ is called the Cauchy function. Function $K(t,s,\varepsilon)$ can be represented as

$$K(t,s,\varepsilon) = \frac{W_n(t,s,\varepsilon)}{W(s,\varepsilon)}, \tag{6}$$

where $W(s,\varepsilon)$ is the Wronskian of the fundamental solution system $y_1(s,\varepsilon),..., y_n(s,\varepsilon)$ for equation (4), and $W_n(t,s,\varepsilon)$ is the n-th order determinant obtained from the Wronskian $W(s,\varepsilon)$ by replacing the n-th row with the fundamental set of solutions $y_1(t,\varepsilon),..., y_n(t,\varepsilon)$.

For the Cauchy function $K(t,s,\varepsilon)$ in view (4), (6), the following asymptotic representation holds as $\varepsilon \to 0$:

$$K^{(j)}(t,s,\varepsilon) = \varepsilon \frac{\overline{W}_{n-1}^{(j)}(t,s)}{\mu(s)\overline{W}(s)} + \varepsilon^{n-1-j} \exp\left(\frac{1}{\varepsilon}\int_s^t \mu(x)dx\right) \frac{\mu^j(t) y_{n0}(t)}{\mu^{n-1}(s) y_{n0}(s)} +$$

$$+ O\left(\varepsilon^2 + \varepsilon^{n-j} \exp\left(\frac{1}{\varepsilon}\int_s^t \mu(x)dx\right)\right), j = 0,...,n-1, \tag{7}$$

where $\mu(t) = -A_1(t) < 0$, $y_{n0}(t)$ is given by (5), the determinant $\overline{W}(s)$ is the Wronskian of the fundamental solution system $y_{10}(s),..., y_{n-1,0}(s)$ of equation (3), and $\overline{W}_{n-1}^{(j)}(t,s)$ is the determinant obtained from $\overline{W}(s)$ by replacing the $(n-1)$th row with $y_{10}(t),..., y_{n-1.0}(t)$.

From (7) we obtain the following asymptotic estimations:

$$\left|K^{(j)}(t,s,\varepsilon)\right| \le C\varepsilon,\ j = \overline{0, n-2},\quad \left|K^{(n-1)}(t,s,\varepsilon)\right| \le C\left(\varepsilon + \exp\left(-\frac{\gamma(t-s)}{\varepsilon}\right)\right), \tag{8}$$

where $C > 0$ is a constant independent of ε.

Let functions $\Phi_i(t,\varepsilon), i = 1,..,n$ are solutions of the following problem:

$$L_\varepsilon \Phi_i(t,\varepsilon) = 0,\ i = 1,...,n, \quad h_k \Phi_i(t,\varepsilon) = \begin{cases} 1, i = k, \\ 0, i \ne k, \end{cases} \quad k = 1,...,n. \tag{9}$$

Functions $\Phi_i(t,\varepsilon), i = 1,...,n$, are called boundary functions and can be represented in the form:

$$\Phi_i(t,\varepsilon) = \frac{\Delta_i(t,\varepsilon)}{\Delta(\varepsilon)}, \quad i = 1,...,n, \tag{10}$$

where $\Delta(\varepsilon) = \begin{vmatrix} h_1 y_1(t,\varepsilon) & ... & h_1 y_n(t,\varepsilon) \\ ... & ... & ... \\ h_n y_1(t,\varepsilon) & ... & h_n y_n(t,\varepsilon) \end{vmatrix}$, $\Delta_i(t,\varepsilon)$ is the determinant obtained from $\Delta(\varepsilon)$ by replacing the i-th row by the system of fundamental solutions $y_1(t,\varepsilon),...,y_n(t,\varepsilon)$ of equation (3). For the determinant $\Delta(\varepsilon)$, using formulas (2) and (4), we obtain the following asymptotic representation as $\varepsilon \to 0$:

$$\Delta(\varepsilon) = \frac{1}{\varepsilon^m}\mu^m(0)\overline{\Delta}(1+O(\varepsilon)), \tag{11}$$

where $\overline{\Delta}$ is the determinant given by the condition (IV).

For the boundary functions $\Phi_i(t,\varepsilon), i=1,...,n$, from (10) in view (4), (11) we obtain asymptotic representation as $\varepsilon \to 0$:

$$\Phi_i^{(j)}(t,\varepsilon) = \frac{\overline{\Delta}_i^{(j)}(t)}{\overline{\Delta}} + (-1)^{i+n}\varepsilon^{m-j}\cdot\frac{y_{n0}(t)\mu^j(t)\overline{\Delta}_i}{\mu^m(0)\overline{\Delta}}\exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)+$$

$$+O\left(\varepsilon + \varepsilon^{m+1-j}\exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\right), \quad i=1,2,...,n, \quad j=0,...,n-1; \tag{12}$$

where $\overline{\Delta}_i^{(j)}(t)$ is the determinant obtained from $\overline{\Delta}$ by replacing the i-th row with the row $y_{10}^{(j)}(t),...,y_{n-1,0}^{(j)}(t),0$, and $\overline{\Delta}_i, i=1,...,n$, are determinants obtained from the matrix $\begin{pmatrix} h_1 y_{10}(t) & ... & h_1 y_{n-1,0}(t) \\ ... & ... & ... \\ h_n y_{10}(t) & ... & h_n y_{n-1,0}(t)\end{pmatrix}$ by deleting the i - th row.

The proof of the (12) follows from (10) if one takes into account (2), (4), and (11).

Then for boundary functions $\Phi_i(t,\varepsilon), i=1,..,n,$ we obtain the following asymptotic estimations as $\varepsilon \to 0$:

$$\left|\Phi_i^{(j)}(t,\varepsilon)\right| \le C + C\varepsilon^{m-j}\exp\left(-\gamma\frac{t}{\varepsilon}\right), \quad j=0,...,n-1; \; i=1,...,n, \tag{13}$$

where $C>0$ is a constant independent of ε.

3. Main results

We seek the solution of the boundary value problem (1) and (2) in the form:

$$y(t,\varepsilon) = C_1\Phi_1(t,\varepsilon) + ... + C_n\Phi_n(t,\varepsilon) + \frac{1}{\varepsilon}\int_0^t K(t,s,\varepsilon)z(s,\varepsilon)ds, \tag{14}$$

where $\Phi_i(t,\varepsilon), i=1,...,n$ are boundary functions and expressed by the formula (10), function $K(t,s,\varepsilon)$ is the Cauchy function can be represented by the formula (6), $C_i, i=1,...,n$ are unknown constants, $z(t,\varepsilon)$ is an unknown function. Substituting (14) into equation (1) we obtain that $z(t,\varepsilon)$ satisfies the following Fredholm integral equation of the second kind:

$$z(t,\varepsilon) = f(t,\varepsilon) + \int_0^1 H(t,s,\varepsilon)z(s,\varepsilon)ds, \tag{15}$$

where

$$f(t,\varepsilon) = F(t) + \sum_{i=1}^n C_i \int_0^1 \sum_{j=0}^{m+1} H_j(t,x)\Phi_i^{(j)}(x,\varepsilon)dx, \tag{16}$$

$$H(t,s,\varepsilon)=\frac{1}{\varepsilon}\int_0^1\sum_{j=0}^{m+1}H_j(t,x)K^{(j)}(x,s,\varepsilon)dx$$

Assume that the following condition is valid.

V. $\lambda=1$ is not an eigenvalue of the kernel $H(t,s,\varepsilon)$.

In view of the condition V integral equation (15) has an unique solution that can be represented in the form

$$z(t,\varepsilon)=\overline{F}(t,\varepsilon)+\sum_{i=1}^{n}C_i\overline{\varphi}_i(t,\varepsilon)\,, \tag{17}$$

where

$$\overline{F}(t,\varepsilon)\equiv F(t)+\int_0^1 R(t,s,\varepsilon)F(s)ds\,,$$

$$\overline{\varphi}_i(t,\varepsilon)=\int_0^1\sum_{j=0}^{m+1}\overline{H}_j(t,x,\varepsilon)\Phi_i^{(j)}(x,\varepsilon)dx,\ i=1,...n\,, \tag{18}$$

$$\overline{H}_j(t,x,\varepsilon)=H_j(t,x)+\int_0^1 R(t,s,\varepsilon)H_j(s,x)ds,\ j=0,...,m+1$$

and $R(t,s,\varepsilon)$ is a resolvent of the kernel $H(t,s,\varepsilon)$ such that

$$R(t,s,\varepsilon)=\overline{R}(t,s)+O(\varepsilon)\,, \tag{19}$$

where $\overline{R}(t,s)$ is a part of $R(t,s,\varepsilon)$ independent of ε. The last representation can be obtained easily by using (9), (16) and boundedness of the kernel $H(t,s,\varepsilon)$ as $\varepsilon\to 0$.

Use (17) in the right-hand side of (14) to obtain the solution of the boundary value problem (1) and (2) in the form

$$y(t,\varepsilon)=\sum_{i=1}^{n}C_i(\varepsilon)Q_i(t,\varepsilon)+P(t,\varepsilon)\,, \tag{20}$$

where

$$Q_i(t,\varepsilon)=\Phi_i(t,\varepsilon)+\frac{1}{\varepsilon}\int_0^t K(t,s,\varepsilon)\overline{\varphi}_i(s,\varepsilon)ds\,,\ P(t,\varepsilon)=\frac{1}{\varepsilon}\int_0^t K(t,s,\varepsilon)\overline{F}(s,\varepsilon)ds \tag{21}$$

Now, we determine unknown constants $C_i(\varepsilon), i=1,...,n$. For determining these constants we substitute (20) into (2). Thus, we need to solve the system of algebraic equations

$$\begin{cases} C_1\ (1+d_{11}(\varepsilon))+C_2\ d_{12}(\varepsilon)+...+C_n\ d_{1n}(\varepsilon)=a_1-e_1(\varepsilon), \\ \quad ... \qquad ... \qquad ... \qquad ... \\ C_1\ d_{n1}(\varepsilon)+C_2\ d_{n2}(\varepsilon)+...+C_n\ (1+d_{nn}(\varepsilon))=a_n-e_n(\varepsilon), \end{cases} \tag{22}$$

where

$$d_{ik}(\varepsilon)=\sum_{j=0}^{r}\frac{\beta_{ij}}{\varepsilon}\int_0^1 K^{(j)}(1,s,\varepsilon)\overline{\varphi}_k(s,\varepsilon)ds,\quad i,k=1,...,n,$$

$$e_i(\varepsilon)=\sum_{j=0}^{r}\frac{\beta_{ij}}{\varepsilon}\int_0^1 K^{(j)}(1,s,\varepsilon)\overline{F}(s,\varepsilon)ds,\ i=1,...,n.$$

In view of (7), (18), (19), we have following asymptotic representations as $\varepsilon \to 0$

$$d_{ik}(\varepsilon) = \overline{d}_{ik} + O(\varepsilon),\ i,k = 1,...,n\ ,\ e_i(\varepsilon) = \overline{e}_i + O(\varepsilon),\ i = 1,...,n\ , \tag{23}$$

where

$$\overline{d}_{ik} = \sum_{j=0}^{r} \beta_{ij} \int_0^1 \frac{\overline{W}_{n-1}^{(j)}(1,s)}{\mu(s)\overline{W}(s)} \overline{H}_k(s)ds,\ \ i,k = 1,...,n, \tag{24}$$

$$\overline{e}_i = \sum_{j=0}^{r} \beta_{ij} \int_0^1 \frac{\overline{W}_{n-1}^{(j)}(1,s)}{\mu(s)\overline{W}(s)} \overline{F}(s)ds,\ i = 1,...,n\ ,$$

the functions $\overline{H}_k(t), \overline{H}_i(t,x)$, and $\overline{F}(t)$ have the form

$$\overline{H}_k(t) \equiv (-1)^{k+n+1} \frac{\overline{\Delta}_k}{\overline{\Delta}} \overline{H}_{m+1}(t,0) + \int_0^1 \sum_{j=0}^{m+1} \overline{H}_j(t,x) \frac{\overline{\Delta}_k^{(i)}(x)}{\overline{\Delta}} dx,\ k = 1,...,n\ ,$$

$$\overline{H}_i(t,x) \equiv H_i(t,x) + \int_0^1 \overline{R}(t.s) H_i(s,x)ds,\ i = 0,...,m+1\ , \tag{25}$$

$$\overline{F}(t) \equiv F(t) + \int_0^1 \overline{R}(t,s)F(s)ds.$$

For the main determinant $\omega(\varepsilon)$ of the system (22) in view (8), (21), (23) we have asymptotic representation as $\varepsilon \to 0$:

$$\omega(\varepsilon) = \overline{\omega} + O(\varepsilon),$$

where $$\overline{\omega} = \begin{vmatrix} 1+\overline{d}_{11} & \overline{d}_{12} & ... & \overline{d}_{1n} \\ \overline{d}_{21} & 1+\overline{d}_{22} & ... & \overline{d}_{2n} \\ ... & ... & ... & ... \\ \overline{d}_{n1} & \overline{d}_{n2} & ... & 1+\overline{d}_{nn} \end{vmatrix}.$$

Assume that the following conditions is valid:

VI. $\overline{\omega} \neq 0$.

Thus, in view of condition I-VI, the singularly perturbed boundary value problem (1) and (2) has a unique solution $y(t,\varepsilon)$, which can be presented in the following form

$$y(t,\varepsilon) = \sum_{i=1}^{n} C_i(\varepsilon) Q_i(t,\varepsilon) + P(t,\varepsilon)\ , \tag{26}$$

where $Q_i(t,\varepsilon), i = 1,...,n$, and $P(t,\varepsilon)$, defined by the formula (21), $C_i(\varepsilon), i = 1,...,n,$ are the solutions of the system (22).

Then the following theorem is valid.

Theorem. If conditions I-VI are fulfilled, then for the solution $y(t,\varepsilon)$ of the boundary value problem (1) and (2), and its derivatives the following asymptotic representations hold as $\varepsilon \to 0$:

$$y^{(q)}(t,\varepsilon)=\sum_{i=1}^{n}\frac{\overline{\omega}_i}{\overline{\omega}}\left[\frac{\overline{\Delta}_i^{(q)}(t)}{\overline{\Delta}}+\int_0^t\frac{\overline{W}_t^{(q)}(t,s)}{\mu(s)\overline{W}(s)}\left(\int_0^1\sum_{j=0}^{m+1}\overline{H}_j(s,x)\frac{\overline{\Delta}_i^{(j)}(x)}{\overline{\Delta}}dx+(-1)^{i+n+1}\frac{\overline{\Delta}_i}{\overline{\Delta}}\overline{H}_{m+1}(s,0)\right)ds+\right.$$

$$\left.+(-1)^{i+n}\varepsilon^{m-q}\frac{y_{n0}(t)\mu^q(t)}{\mu^m(0)}\frac{\overline{\Delta}_i}{\overline{\Delta}}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)\right]+\int_0^t\frac{\overline{W}_t^{(q)}(t,s)}{\mu(s)\overline{W}(s)}\overline{F}(s)ds+O\left(\varepsilon+\varepsilon^{m+1-q}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)\right),$$

$$q=0,...m,$$

$$y^{(q)}(t,\varepsilon)=\sum_{i=1}^{n}\frac{\overline{\omega}_i}{\overline{\omega}}\left[\frac{\overline{\Delta}_i^{(q)}(t)}{\overline{\Delta}}+\int_0^t\frac{\overline{W}_t^{(q)}(t,s)}{\mu(s)\overline{W}(s)}\left(\int_0^1\sum_{j=0}^{m+1}\overline{H}_j(s,x)\frac{\overline{\Delta}_i^{(j)}(x)}{\overline{\Delta}}dx+(-1)^{i+n+1}\frac{\overline{\Delta}_i}{\overline{\Delta}}\overline{H}_{m+1}(s,0)\right)ds+\right.$$

$$\left.+(-1)^{i+n}\frac{y_{n0}(t)\mu^q(t)}{\varepsilon^{q-m}\mu^m(0)}\frac{\overline{\Delta}_i}{\overline{\Delta}}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)\right]+\int_0^t\frac{\overline{W}_t^{(q)}(t,s)}{\mu(s)\overline{W}(s)}\overline{F}(s)ds+O\left(\varepsilon+\frac{1}{\varepsilon^{q-m-1}}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)\right),$$

$$q=m+1,...n-2, \tag{27}$$

$$y^{(n-1)}(t,\varepsilon)=\sum_{i=1}^{n}\frac{\overline{\omega}_i}{\overline{\omega}}\left[\frac{\overline{\Delta}_i^{(n-1)}(t)}{\overline{\Delta}}+\int_0^t\frac{\overline{W}_t^{(n-1)}(t,s)}{\mu(s)\overline{W}(s)}\left(\int_0^1\sum_{j=0}^{m+1}\overline{H}_j(s,x)\frac{\overline{\Delta}_i^{(j)}(x)}{\overline{\Delta}}dx+(-1)^{i+n+1}\frac{\overline{\Delta}_i}{\overline{\Delta}}\overline{H}_{m+1}(s,0)\right)ds+\right.$$

$$\left.+(-1)^{i+n}\varepsilon^{m-n+1}\frac{y_{n0}(t)\mu^{n-1}(t)}{\mu^m(0)}\frac{\overline{\Delta}_i}{\overline{\Delta}}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)\right]+\int_0^t\frac{\overline{W}_t^{(n-1)}(t,s)}{\mu(s)\overline{W}(s)}\overline{F}(s)ds-\frac{\overline{F}(t)}{\mu(t)}+$$

$$+\frac{y_{n0}(t)\mu^{n-1}(t)}{\varepsilon^{n-1-m}\mu^n(0)}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)+O\left(\varepsilon+\frac{1}{\varepsilon^{n-2-m}}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)\right).$$

Proof. By means of (7), (12), (16), (18), (19) and (25), we obtain the following asymptotic representation as $\varepsilon\to 0$ for functions $Q_i(t,\varepsilon), i=1,...,n$, and $P^{(q)}(t,\varepsilon),\ q=0,...,n-1$, defined in (21):

$$Q_i^{(q)}(t,\varepsilon)=\frac{\overline{\Delta}_i^{(q)}(t)}{\overline{\Delta}}+\int_0^t\frac{\overline{W}_t^{(q)}(t,s)}{\mu(s)\overline{W}(s)}\left[\int_0^1\sum_{j=0}^{m+1}\overline{H}_j(s,x)\frac{\overline{\Delta}_i^{(j)}(x)}{\overline{\Delta}}dx+(-1)^{i+n+1}\frac{\overline{\Delta}_i}{\overline{\Delta}}\overline{H}_{m+1}(s,0)\right]ds+$$

$$+(-1)^{i+n}\varepsilon^{m-q}\frac{y_{n0}(t)\mu^q(t)}{\mu^m(0)}\frac{\overline{\Delta}_i}{\overline{\Delta}}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)+O\left(\varepsilon+\varepsilon^{m+1-q}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)\right),$$

$$i=1,...,n,\ q=0,...n-1, \tag{28}$$

$$P^{(q)}(t,\varepsilon)=\int_0^t\frac{\overline{W}_t^{(q)}(t,s)}{\mu(s)\overline{W}(s)}\overline{F}(s)ds+O(\varepsilon),\ \ q=0,...,n-2,$$

$$P^{(n-1)}(t,\varepsilon)=\int_0^t\frac{\overline{W}_t^{(n-1)}(t,s)}{\mu(s)\overline{W}(s)}\overline{F}(s)ds-\frac{\overline{F}(t)}{\mu(t)}+\frac{y_{n0}(t)\mu^{n-1}(t)}{\mu^n(0)}\exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)+O(\varepsilon)$$

In view of (23), (24) $C_i(\varepsilon)$ can be asymptotically represented in the form

$$C_i(\varepsilon)=\frac{\overline{\omega}_i}{\overline{\omega}}+O(\varepsilon), i=1,...,n, \tag{29}$$

where $\overline{\omega}_i$ is the determinant obtained from $\overline{\omega}$ by substituting the i-th column by $\begin{pmatrix} a_1-\overline{e}_1 \\ .... \\ a_n-\overline{e}_n \end{pmatrix}$.

From (20) in view (7), (12), (28), (29) we obtain asymptotic representation (27). Theorem is proved.

From asymptotic representation (27) it follows that the solution of the boundary value problem (1) and (2) at the point t=0 has the pole of ε:

$$y^{(j)}(0,\varepsilon)=O(1), j=\overline{0,m}, \; y^{(m+j)}(0,\varepsilon)=O\left(\frac{1}{\varepsilon^j}\right), j=\overline{1,n-1-m}, \quad \varepsilon\to 0 .$$

In this case, we say that the solution of the boundary value problem (1) and (2) at the point $t=0$ has the phenomenon of the m **-**th order initial jump.

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АСИМПТОТИЧЕСКОЕ ПРЕДСТАВЛЕНИЕ СИНГУЛЯРНО ВОЗМУЩЕННЫХ КРАЕВЫХ ЗАДАЧ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Аннотация. Рассмотрим краевую задачу по двум точкам для сингулярно возмущенного линейного интегро-дифференциального уравнения высшего порядка. Асимптотические представления решения по малому параметру получены. Показано, что решение имеет m-го начального скачка в точке $t = 0$.

Ключевые слова: сингулярное возмущенние, интегро-дифференциальное уравнение, малый параметр, начальный скачок.

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СИНГУЛЯРЛЫ АУЫТҚЫҒАН ИНТЕГРАЛДЫ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУГЕ АРНАЛҒАН ШЕКАРАЛЫҚ ЕСЕПТІҢ АСИМТОТИКАЛЫҚ БЕЙНЕЛЕУІ

Аннотация. Біз сингулярлы ауытқыған жоғары ретті интегралды-дифференциалдық теңдеуге арналған екі нүктелі шекаралық есеп қарастырымыз. Шешімінің кіші параметрмен асимтотикалық жіктелуі алынады. Бұл жерде шешімнің $t = 0$ нүктесінде m **-**шы ретті бастапқы секірісі бар екені көрсетілген.

Кілт сөздер: сингулярлы ауытқу, интеграды-дифференциалдық теңдеулер, кіші параметр, бастапқы секіріс.

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