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ON SQUARE ROOT OF STURM-LIUVILLE OPERATOR

Abstract. In this paper, we find square root of the Sturm – Liouville operator and show that this root is a functional-differential operator of first-order. Form of the corresponding boundary problem of this functional - differential equation is found. As a suggestive idea, we use one Putnam theorem. Boundary value conditions of the Sturm-Liouville operator have a very special form, and they are dictated by the method of investigation. The found unitary operator generalizes the known momentum operator.

Keywords. Sturm-Liouville operator, square root of operator, functional differential operator, equations with deviating argument, Kato hypothesis, Macintosh example, Gours operator, inverse problem, spectrum, eigenvalues, eigenfunctions, unitary operator, similarity operator.

1. Introduction. It is known [1.p.393], that if A is a self-adjoint and non-negative operator in a Hilbert space H, then there exists unit self-adjoint operator $B \geq 0$ such that $B^2 = A$. The following theorem from the same source says that not every operator has a square root.

Theorem 1.1 [1.p.357]. Let D be a bounded open set in $\mathbb{C}$ such that the set

$$\Omega = \{\alpha \in \mathbb{C}: \alpha^2 \in D\}$$

is connected and the point 0 does not belong to the closure of the set D. Let H be a set of all holomorphic functions f in D such that

$$\int_D |f|^2 dm_2 < \infty$$

(where m_2 is a Lebesgue flat measure). We give in H scalar product by the formula:

$$(f,g) = \int_D f\bar{g}dm_2.$$

Then H is a Hilbert space. We define the product operator $M \in \mathcal{B}(H)$, assuming

$$(Mf)(z) = zf(z), \qquad (f \in H, z \in D).$$

Then the operator M is invertible in $\mathcal{B}(H)$, but it does not have a square root.

Extending the concept of a root to dissipative operators, Kato hypothesis has been arisen, consisting in the fact that the domain of a root from an operator always coincides with the domain of a root from an adjoint operator. However, in 1972 A.Makintosh [3] built a counterexample, since then the hypothesis was slightly reformulated: find the largest class of operators that satisfies this condition, and very active research in this direction is currently cited [4-57].

Many operators of theoretical physics have square roots [57–64]; in particular, the square root of operator A in a Banach space was found in [65, pp.169-176]. We give an excerpt from this work.

We consider a generating operator A in a Banach space $\mathcal{B}$, that has the following properties:

1) Operator $(I+\gamma^2 A)^{-1}$ exists, is defined everywhere in $\mathcal{B}$ and bounded by one;
2) Operator A^{-1} exists;
3) $\|e^{iAt}\| \leq M,\ -\infty < t < +\infty$.

In these conditions the following lemmas hold.

Lemma 1.1. Operator

$$T = \frac{2e^{-i\pi/4}}{\sqrt{\pi}} A \int_0^{\infty} e^{iAx^2} dx$$

exists as an operator in $\mathcal{B}$ in the domain $D(A)$.

Lemma 1.2. For any $y \in D(A)$ the following equality is true

$$T^2 g = Ag.$$

Due to these results, the following problem arises.

1. **Formulation of the problem.** Find a square root of the Sturm - Liouville operator

$$Ly = -y''(x),\ x \in (0,1), \tag{1.1}$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ \alpha y'(1) + \beta y'(0) = 0, \end{cases} \tag{1.2}$$

where $\alpha,\ \beta$ are arbitrary (yet) complex numbers, satisfying the condition

$$|\alpha| + |\beta| \neq 0. \tag{1.3}$$

2. **Research methods.**

Calculate minors of the boundary matrix

$$\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \beta & 0 & \alpha \end{pmatrix},$$

$$J_{12} = \alpha\beta, \quad J_{13} = 0, \quad J_{14} = \alpha^2, \quad J_{23} = -\beta^2, \quad J_{24} = 0, \quad J_{34} = \alpha\beta.$$

If $J_{14} + J_{32} = \alpha^2 + \beta^2 \neq 0$, then the Sturm - Liouville problem (1.1) - (1.3) has a complete system of eigen and associated functions, see [66., p.41].

Find eigenfunctions of the Sturm-Liouville problem (1.1) - (1.2). General solution of the equation (1.1) has the form:

$$y(x,\lambda) = A\cos\lambda x + B\frac{\sin\lambda x}{\lambda}, \tag{1.4}$$

where $A,\ B$ are arbitrary constants. Putting (1.4) into (1.2), we have

$$y'(x,\lambda) = -\lambda A\sin\lambda x + B\cos\lambda x,$$

$$y(0) = A, \qquad y'(0) = B, \qquad y(1) = A\cos\lambda + B\frac{\sin\lambda}{\lambda},$$

$$y'(1) = -\lambda A\sin\lambda + B\cos\lambda;$$

$$\begin{cases} A\cdot\alpha + \beta\left(A\cos\lambda + B\dfrac{\sin\lambda}{\lambda}\right) = 0, \\ \alpha(-\lambda A\sin\lambda + \beta\cos\lambda) + B\cdot\beta = 0; \end{cases}$$

$$\begin{cases} A(\alpha + \beta\cos\lambda) + B\cdot\dfrac{\beta\sin\lambda}{\lambda} = 0, \\ A(-\lambda\alpha\sin\lambda) + B(\alpha\cos\lambda + \beta) = 0. \end{cases}$$

Therefore, we obtained the system of equations

$$\begin{cases} A(\alpha + \beta\cos\lambda) + B\cdot\dfrac{\beta\sin\lambda}{\lambda} = 0, \\ A(-\lambda\alpha\sin\lambda) + B(\alpha\cos\lambda + \beta) = 0. \end{cases}$$

determinant of which has the form

$$\Delta(\lambda)=\begin{vmatrix}\alpha+\beta\cos\lambda & \frac{\beta\sin\lambda}{\lambda}\\ -\lambda\alpha\sin\lambda & \alpha\cos\lambda+\beta\end{vmatrix}=(\alpha+\beta\cos\lambda)(\alpha\cos\lambda+\beta)+\alpha\beta sin^2\lambda=$$

$$=\alpha^2\cos\lambda+\alpha\beta+\beta\alpha cos^2\lambda+\beta^2\cos\lambda+\alpha\beta sin^2\lambda=$$

$$=\alpha^2\cos\lambda+\beta^2\cos\lambda+2\alpha\beta=(\alpha^2+\beta^2)\cos\lambda+2\alpha\beta.$$

If $\Delta(\lambda)=0$, then $\cos\lambda=-\frac{2\alpha\beta}{\alpha^2+\beta^2}$,

$$\dot{\Delta}(\lambda)=-(\alpha^2+\beta^2)\sin\lambda=\mp(\alpha^2+\beta^2)\sqrt{1-\frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2)^2}}=$$

$$=\mp(\alpha^2+\beta^2)\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}=\mp(\alpha^2-\beta^2).$$

Consequently, if $\alpha^2-\beta^2\neq 0$, then the associated functions of the Sturm – Liouville operator are absent, so the eigenfunctions of the boundary value problem (1.1) - (1.2) are complete in the space $L_2(0,1)$.

Assuming,

$$A=\frac{\beta\sin\lambda}{\lambda},\ B=-(\alpha+\beta\cos\lambda)$$

and taking into account that

$$\cos\lambda=-\frac{2\alpha\beta}{\alpha^2+\beta^2}$$

we have

$$B=-(\alpha+\beta\cos\lambda)=-\left(\alpha-\frac{2\alpha\beta^2}{\alpha^2+\beta^2}\right)=-\alpha\left(1-\frac{2\beta^2}{\alpha^2+\beta^2}\right)=$$

$$=-\alpha\cdot\frac{\alpha^2+\beta^2-2\beta^2}{\alpha^2+\beta^2}=-\alpha\cdot\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2};$$

$$A=\pm\frac{\beta}{\lambda}\sqrt{1-\frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2)^2}}=\pm\frac{\beta}{\lambda}\cdot\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}.$$

If $\lambda=0$, then $\Delta(0)=\alpha^2+\beta^2+2\alpha\beta=(\alpha+\beta)^2$, consequently, if $\alpha^2-\beta^2\neq 0$, then $\Delta(0)\neq 0$.

If $|A|+|B|=0$, then $|\alpha|\cdot|(\alpha^2-\beta^2)|+|\beta|\cdot|(\alpha^2-\beta^2)|=0,=>$

$(|\alpha|+|\beta|)|\alpha^2-\beta^2|=0$, hence by the condition (1.3) it follows that

$$\alpha^2-\beta^2=0.$$

Therefore, when $\alpha^2-\beta^2\neq 0$ the solution (eigenfunction)

$$y(x,\lambda)=A\cos\lambda x+B\frac{\sin\lambda x}{\lambda}$$

is not degenerate, i.e. $y(x,\lambda)\not\equiv 0$.

Further, we transform the eigenfunction:

$$y(x,\lambda)=\pm\frac{\beta}{\lambda}\cdot\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\cos\lambda x-\frac{\alpha(\alpha^2-\beta^2)}{\lambda(\alpha^2+\beta^2)}\sin\lambda x=$$

$$=\frac{\alpha^2-\beta^2}{\lambda(\alpha^2+\beta^2)}(\pm\beta\cos\lambda x-\alpha\sin\lambda x);$$

$$y(1-x,\lambda)=\frac{\alpha^2-\beta^2}{\lambda(\alpha^2+\beta^2)}[\pm\beta\cos\lambda(1-x)-\alpha\sin\lambda(1-x)]=$$

$$= \frac{\alpha^2 - \beta^2}{\lambda(\alpha^2 + \beta^2)}(\pm\beta \cos\lambda \cos\lambda x \pm \beta \sin\lambda \sin\lambda x - \alpha \sin\lambda \cos\lambda x +$$

$$+\alpha \cos\lambda \sin\lambda x) = \frac{\alpha^2 - \beta^2}{\lambda(\alpha^2 + \beta^2)}[(\pm\beta \cos\lambda - \alpha \sin\lambda) \cos\lambda x +$$

$$+(\alpha \cos\lambda \pm \beta \sin\lambda) \cdot \sin\lambda x];$$

Further,

$$\pm\beta \cos\lambda - \alpha \sin\lambda = \mp \frac{2\alpha\beta^2}{\alpha^2 + \beta^2} \mp \frac{\alpha(\alpha^2 - \beta^2)}{\alpha^2 + \beta^2} = \frac{\mp\alpha^3 \pm \alpha\beta^2 \mp 2\alpha\beta^2}{\alpha^2 + \beta^2} =$$

$$= \frac{\mp\alpha^3 \mp \alpha\beta^2}{\alpha^2 + \beta^2} = \mp \frac{\alpha(\alpha^2 + \beta^2)}{\alpha^2 + \beta^2} = \mp\alpha$$

$$\alpha \cos\lambda \pm \beta \sin\lambda = -\frac{2\alpha^2\beta}{\alpha^2 + \beta^2} \pm \beta\left(\pm\frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}\right) =$$

$$= \frac{\beta(\alpha^2 - \beta^2) - 2\alpha^2\beta}{\alpha^2 + \beta^2} = \frac{\beta\alpha^2 - \beta^3 - 2\alpha^2\beta}{\alpha^2 + \beta^2} = \frac{-\beta^3 - \alpha^2\beta}{\alpha^2 + \beta^2} =$$

$$= -\frac{\beta(\alpha^2 + \beta^2)}{\alpha^2 + \beta^2} = -\beta.$$

Consequently,

$$y(1 - x, \lambda) = \frac{\alpha^2 - \beta^2}{\lambda(\alpha^2 + \beta^2)}(\mp\alpha \cos\lambda x - \beta \sin\lambda x) =$$

$$= \mp \frac{\alpha^2 - \beta^2}{\lambda(\alpha^2 + \beta^2)}(\alpha \cos\lambda x \pm \beta \sin\lambda x); \quad (1.5)$$

$$y'(x, \lambda) = \frac{\alpha^2 - \beta^2}{\lambda(\alpha^2 + \beta^2)} \cdot \lambda(\mp\beta \sin\lambda x - \alpha \cos\lambda x) =$$

$$-\lambda \frac{\alpha^2 - \beta^2}{\lambda(\alpha^2 + \beta^2)}(\alpha \cos\lambda x \pm \beta \sin\lambda x) =$$

$$= -\frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}(\alpha \cos\lambda x \pm \beta \sin\lambda x). \quad (1.6)$$

Comparison of the formulas (1.5) - (1.6) shows that

$$y'(x, \lambda) = \pm\lambda y(1 - x, \lambda). \quad (1.7)$$

We consider the case $\alpha^2 - \beta^2 = 0$ separately.

If $\alpha^2 - \beta^2 = 0$, then $\beta = \pm\alpha$,

$$\Delta(\lambda) = 2\alpha^2 \cos\lambda \pm 2\alpha^2 = 2\alpha^2(\cos\lambda \pm 1), => \cos\lambda \pm 1 = 0.$$

In this case,

$$B = -(\alpha + \beta \cos\lambda) = -(\alpha \pm \alpha \cos\lambda) = -\alpha(1 \pm \cos\lambda) = 0,$$

$$A = \beta \frac{\sin\lambda}{\lambda} = 0.$$

In this case we should choose the coefficients by using another method.

Let

$y(x,\lambda) = \cos\lambda\, x + \sin\lambda\, x$, where $\cos\lambda \pm 1 = 0$, then

$$W[y(x,\lambda), y(1-x,\lambda)] = \begin{vmatrix} \cos\lambda\, x + \sin\lambda\, x & \cos\lambda\, x - \sin\lambda\, x \\ \lambda(\cos\lambda\, x - \sin\lambda\, x) & -\lambda(\cos\lambda\, x + \sin\lambda\, x) \end{vmatrix} =$$

$$= -\lambda[(\cos\lambda\, x + \sin\lambda\, x)^2 + (\cos\lambda\, x - \sin\lambda\, x)^2] =$$

$$= -\lambda(1+1) = -2\lambda \neq 0.$$

$$y(1-x,\lambda) = \cos\lambda\,(1-x) + \sin\lambda\,(1-x) = \cos\lambda\cos\lambda\, x + \sin\lambda\sin\lambda\, x +$$

$$+ \sin\lambda\cos\lambda\, x - \cos\lambda\sin\lambda\, x = \cos\lambda\,(\cos\lambda\, x - \sin\lambda\, x);$$

$$y'(x,\lambda) = \lambda(-\sin\lambda\, x + \cos\lambda\, x) = \lambda(\cos\lambda\, x - \sin\lambda\, x) = \frac{\lambda}{\cos\lambda} y(1-x,\lambda),$$

where $\cos\lambda = \pm 1$, consequently,

$$y'(x,\lambda) = \pm\lambda y(1-x,\lambda). \tag{1.7}$$

Let's check the boundary conditions:

$$y(x,\lambda) = \frac{\alpha^2 - \beta^2}{\lambda(\alpha^2+\beta^2)}(\mp\beta\cos\lambda x - \alpha\sin\lambda\, x) =$$

$$= \frac{K}{\lambda}(\mp\beta\cos\lambda x - \alpha\sin\lambda\, x).$$

$$y(x,-\lambda) = \frac{\alpha^2 - \beta^2}{-\lambda(\alpha^2+\beta^2)}(\mp\beta\cos\lambda x + \alpha\sin\lambda\, x) =$$

$$= \frac{K}{-\lambda}(\mp\beta\cos\lambda x + \alpha\sin\lambda\, x),$$

$$y'(x,\lambda) = K(\mp\beta\sin\lambda x - \alpha\cos\lambda x),$$

$$y'(x,-\lambda) = K(\pm\sin\lambda x - \alpha\cos\lambda x),$$

$$W[y(x,\lambda), y(x,-\lambda)] = \begin{vmatrix} \pm\beta\frac{K}{\lambda} & \mp\beta\frac{K}{\lambda} \\ -\alpha K & -\alpha K \end{vmatrix} = -\alpha K \begin{vmatrix} \pm\beta & \mp\beta \\ 1 & 1 \end{vmatrix} \frac{K}{\lambda} =$$

$$= -\frac{\alpha K^2}{\lambda}(\pm\beta \pm \beta) = \mp\frac{2\alpha\beta K^2}{\lambda} = \mp\frac{2\alpha\beta}{\lambda}\left[\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\right]^2.$$

Lemma 2.1. If

$$\frac{\alpha\beta}{\lambda} \cdot \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2} \neq 0$$

then any eigenfunction of the Sturm-Liouville boundary value problem (1.1) - (1.2)

$$Ly = -y''(x) = \lambda^2 y(x),\ \ x \in (0,1), \tag{1.1}$$

$$\begin{cases} \alpha y(0) + \beta y(1) = 0, \\ \alpha y'(1) + \beta y'(0) = 0, \end{cases} \tag{1.2}$$

is an eigenfunction of the boundary value problem

$$ly = y'(1-x) = \lambda y(x),$$

$$\alpha y(0) + \beta y(1) = 0;$$

i.e. the formula

$$l^2 y = \left(S\frac{d}{dx}\right)^2 y = Ly = -y''(x)$$

holds, where the operator S has the following form

$$Su(x) = u(1-x), \qquad \forall u(x) \in L^2(0,1).$$

In this sense, the operator L is the square root of the Sturm-Liouville operator.

The case $\alpha \cdot \beta(\alpha^2 - \beta^2) = 0$ requires separate study.

We find the operator $\sqrt{L}$, where

$$Ly = -y''(x),\ x \in (0,1),$$

$$y(0) - y(1) = 0,\ y'(0) - y'(1) = 0.$$

General idea of the solution is as follows: first, we represent the operator L in the form

$$L = l^+ l$$

the product of two mutually conjugate operators, then we find the similarity operator T such that

$$Tl^+ l = ll^+ T.$$

From Putnam's theorem [1. p.337] it follows that such an operator will certainly be unitary, that is, there is the equality

$$T^* T = TT^* = I,$$

where I is unit operator. In our observation [67], our desired operator has the form Tl, where

$$ly(x) = y'(x), \qquad y(0) - y(1) = 0.$$

We solve these problems step by steps, and more in detail.

Let $$ly(x) = y'(x),\ x \in (0,1),$$

$$\alpha y(0) + \beta y(1) = 0,\ |\alpha| + |\beta| \neq 0. \tag{1.8}$$

We find formal conjugate operator l^+.

Let $y \in D(l)$ and $z \in D(l^+)$, then the formula

$$(ly, z) = (y, l^+ z),$$

holds, where the scalar product has the form

$$(u,v)=\int_0^1 u(x)\overline{v(x)}\,dx.$$

$$(ly,z)=\int_0^1 y'(x)\bar{z}(x)dx=\int_0^1 \bar{z}(x)\,dy=\overline{z(x)}y(x)\Big|_0^1-\int_0^1 y(x)\bar{z}'(x)dx,$$

thus, we have $\overline{z(1)}y(1)-\overline{z(0)}y(0)=0$. Combining this condition with the boundary condition (1.8), we obtain the system of equations

$$\begin{cases}\overline{z(0)}y(0)-\overline{z(1)}y(1)=0,\\ \alpha y(0)+\beta y(1)=0,\end{cases}$$

which has nontrivial solution, therefore

$$\Delta=\begin{vmatrix}\overline{z(0)} & -\overline{z(1)}\\ \alpha & \beta\end{vmatrix}=\beta\overline{z(0)}+\alpha\overline{z(1)}=0,$$

then

$$\bar{\beta}z(0)+\bar{\alpha}z(1)=0.$$

Therefore,

$$l^{+}z=-z'(x),\qquad x\in(0,1),$$
$$\bar{\beta}z(0)+\bar{\alpha}z(1)=0.$$

Remark. If $D(l)=D(l^{+})$,

then

$$\begin{cases}\alpha y(0)+\beta y(1)=0,\\ \bar{\beta}y(0)+\bar{\alpha}y(1)=0;\end{cases}\;=>\;|\alpha|^2-|\beta|^2=0,\qquad |\alpha|=|\beta|.$$

We find the operator $l^{+}l$, and research its spectral properties.

Assuming $A=l^{+}l$, we have

$$Ay=l^{+}ly=l^{+}y'=-y''(x),\qquad x\in(0,1);$$
$$\alpha y(0)+\beta y(1)=0,\qquad \bar{\beta}y'(0)+\bar{\alpha}y'(1)=0.$$

We construct the boundary matrix

$$\begin{pmatrix}\alpha & 0 & \beta & 0\\ 0 & \bar{\beta} & 0 & \bar{\alpha}\end{pmatrix}$$

and calculate minors:

$$J_{12}=\alpha\bar{\beta},\qquad J_{13}=0,\qquad J_{14}=|\alpha|^2,\qquad J_{23}=-|\beta|^2,$$
$$J_{24}=0,\qquad J_{34}=\beta\bar{\alpha}.$$

We find the characteristic function [66, p.35].

$$\Delta(\lambda)=J_{12}+J_{34}+(J_{14}+J_{32})\cos\lambda+J_{13}\frac{\sin\lambda}{\lambda}+J_{24}\lambda\sin\lambda=$$
$$=\alpha\bar{\beta}+\beta\bar{\alpha}+(|\alpha|^2+|\beta|^2)\cos\lambda=0,$$

$$\cos\lambda=-\frac{\alpha\bar{\beta}+\beta\bar{\alpha}}{|\alpha|^2+|\beta|^2},=>\;\lambda_n=arccos\left[-\frac{\alpha\bar{\beta}+\beta\bar{\alpha}}{|\alpha|^2+|\beta|^2}\right]+2n\pi,n=0,\pm1,\pm2,..$$

Now we find the eigenfunctions:

$$Ay=-y''(x)=\lambda^2y(x),\;x\in(0,1);$$

General solution of this equation has the form

$$y(x,\lambda) = A\cos\lambda x + B\frac{\sin\lambda x}{\lambda},$$

thus we have,

$$y(0) = A, \qquad y(1) = A\cos\lambda + B\frac{\sin\lambda}{\lambda},$$

$$\alpha A + \beta\left(A\cos\lambda + B\frac{\sin\lambda}{\lambda}\right) = 0,$$

$$A(\alpha + \beta\cos\lambda) + \beta\frac{\sin\lambda}{\lambda}\cdot B = 0;$$

Further,

$$y'(x) = -\lambda A\sin\lambda x + B\cos\lambda x,$$

$$y'(0) = B, \qquad y'(1) = -\lambda A\sin\lambda + B\cos\lambda,$$

$$\bar{\beta}B + \bar{\alpha}(-\lambda A\sin\lambda + B\cos\lambda) = 0,$$

$$-\lambda\bar{\alpha}\sin\lambda\, A + B(\bar{\alpha}\cos\lambda + \bar{\beta}) = 0.$$

We construct the system of equations:

$$\begin{cases} A(\alpha + \beta\cos\lambda) + B\cdot\beta\frac{\sin\lambda}{\lambda} = 0, \\ A(-\lambda\bar{\alpha}\sin\lambda) + B(\bar{\alpha}\cos\lambda + \bar{\beta}) = 0. \end{cases}$$

Calculate determinant of the system:

$$\Delta = \begin{vmatrix} \alpha + \beta\cos\lambda & \beta\frac{\sin\lambda}{\lambda} \\ -\lambda\bar{\alpha}\sin\lambda & \bar{\alpha}\cos\lambda + \bar{\beta} \end{vmatrix} = 0.$$

$$(\alpha + \beta\cos\lambda)(\bar{\alpha}\cos\lambda + \bar{\beta}) + \bar{\alpha}\beta\sin^2\lambda = |\alpha|^2\cos\lambda + \alpha\bar{\beta} + \beta\bar{\alpha}\cos^2\lambda +$$

$$+|\beta|^2\cos\lambda + \bar{\alpha}\beta\sin^2\lambda = (|\alpha|^2 + |\beta|^2)\cos\lambda + \alpha\bar{\beta} + \bar{\alpha}\beta = 0,$$

$$\cos\lambda = -\frac{\alpha\bar{\beta} + \bar{\alpha}\beta}{|\alpha|^2 + |\beta|^2}, \Rightarrow\ \lambda_n = arccos\left[-\frac{\alpha\bar{\beta} + \bar{\alpha}\beta}{|\alpha|^2 + |\beta|^2}\right] + 2n\pi, n = 0, \pm 1, \pm 2, ..$$

Assuming

$$A_n = \beta\frac{\sin\lambda_n}{\lambda_n}\cdot K_n, \qquad B_n = -(\alpha + \beta\cos\lambda_n)\cdot K_n,$$

we construct the eigenfunctions:

$$y_n(x) = \frac{K_n}{\lambda_n}[\beta\sin\lambda_n\ \cos\lambda_n x - (\alpha + \beta\cos\lambda_n)\sin\lambda_n\ x] =$$

$$= \frac{K_n}{\lambda_n}(\beta\sin\lambda_n\cos\lambda_n x - \beta\cos\lambda_n\sin\lambda_n x - \alpha\sin\lambda_n x) =$$

$$= \frac{K_n}{\lambda_n}[\beta\sin(\lambda_n - \lambda_n x) - \alpha\sin\lambda_n x] =$$

$$= \frac{K_n}{\lambda_n}[\beta\sin\lambda_n(1 - x) - \alpha\sin\lambda_n x];$$

We formulate the obtained result in the form of Lemma 2.2.

Lemma 2.2. Eigenvalues and eigenfunctions of the operator A have the following form:

a) $\lambda_n = arccos\left[-\frac{\alpha\bar{\beta}+\bar{\alpha}\beta}{|\alpha|^2+|\beta|^2}\right] + 2n\pi, n = 0, \pm 1, \pm 2, \dots;$

b) $y_n(x) = \frac{K_n}{\lambda_n}[\beta \sin \lambda_n (1-x) - \alpha \sin \lambda_n x],$

where K_n are arbitrary constants.

We find the operator $B = ll^+$, and study its spectral properties.

$$ly = y'(x), \;\; x \in (0,1);$$

$$\alpha y(0) + \beta y(1) = 0,$$

$$l^+ z = -z'(x), \qquad x \in (0,1),$$

$$\bar{\beta} z(0) + \bar{\alpha} z(1) = 0.$$

$$l^+ z = -z'(x) \in D(l), => \; \alpha[-z'(0)] + \beta[-z'(1)] = 0,$$

$$Bz = ll^+ z = -z''(x), \qquad x \in (0,1),$$

$$\bar{\beta} z(0) + \bar{\alpha} z(1) = 0, \qquad \alpha z'(0) + \beta z'(1) = 0.$$

We find eigenvalues and eigenfunctions of the operator B:

$$Bz = -z''(x) = \mu^2 z(x), \qquad x \in (0,1);$$

$$\begin{cases} \bar{\beta} z(0) + \bar{\alpha} z(1) = 0, \\ \alpha z'(0) + \beta z'(1) = 0. \end{cases}$$

General solution of the equation $-z''(x) = \mu^2 z(x)$ has the following form:

$$z(x) = A \cos \mu x + B \frac{\sin \mu x}{\mu},$$

where A, B are arbitrary constants, hence we have

$$z(0) = A, \qquad z(1) = A \cos \mu + B \frac{\sin \mu}{\mu},$$

$$\bar{\beta} A + \bar{\alpha}\left(A \cos \mu + B \frac{\sin \mu}{\mu}\right) = 0,$$

$$A(\bar{\beta} + \bar{\alpha} \cos \mu) + \alpha \frac{\sin \mu}{\mu} \cdot B = 0;$$

In a similar way we get:

$$z'(x) = -\mu A \sin \mu x + B \cos \mu x,$$

$$z'(0) = B, \qquad z'(1) = -\mu A \sin \mu + B \cos \mu,$$

$$\alpha B + \beta(-\mu A \sin \mu + B \cos \mu) = 0,$$

$$A(-\beta \mu \sin \mu) + B(\alpha + \beta \cos \mu) = 0.$$

We construct the system of equations:

$$\begin{cases} (\bar{\beta} + \bar{\alpha}\cos\mu)A + \bar{\alpha}\dfrac{\sin\mu}{\mu}\cdot B = 0, \\ (-\beta\mu\sin\mu)A + (\alpha + \beta\cos\mu)B = 0. \end{cases}$$

Calculate determinant of this system of equations:

$$\Delta = \begin{vmatrix} \bar{\beta} + \bar{\alpha}\cos\mu & \bar{\alpha}\dfrac{\sin\mu}{\mu} \\ -\beta\mu\sin\mu & \alpha + \beta\cos\mu \end{vmatrix} = (\bar{\beta} + \bar{\alpha}\cos\mu)(\alpha + \beta\cos\mu) +$$

$$+\bar{\alpha}\beta sin^2\mu = \bar{\beta}\alpha + |\beta|^2\cos\mu + |\alpha|^2\cos\mu + \bar{\alpha}\beta\cos^2\mu + \bar{\alpha}\beta\sin^2\mu =$$

$$= (|\alpha|^2 + |\beta|^2)\cos\mu + \alpha\bar{\beta} + \bar{\alpha}\beta = 0,$$

$$\cos\mu = -\frac{\alpha\bar{\beta} + \bar{\alpha}\beta}{|\alpha|^2 + |\beta|^2}, \Rightarrow \ \mu_n = arccos\left[-\frac{\alpha\bar{\beta} + \bar{\alpha}\beta}{|\alpha|^2 + |\beta|^2}\right] + 2n\pi, n = 0, \pm 1, \pm 2, ..$$

We find the eigenfunctions:

$$A_n = K_n \cdot (\alpha + \beta\cos\mu_n), \qquad B_n = K_n\beta \cdot \mu_n \sin\mu_n,$$

$$z_n(x) = K_n(\alpha + \beta\cos\mu_n)\cos\mu_n x + K_n\beta\cdot\mu_n\sin\mu_n\cdot\frac{\sin\mu}{\mu} =$$

$$= K_n(\alpha\cos\mu_n x + \beta\cos\mu_n\cos\mu_n x + \beta\sin\mu_n\sin\mu_n x) =$$

$$= K_n[\alpha\cos\mu_n x + \beta\cos(\mu_n - \mu_n x)] =$$

$$= K_n[\alpha\cos\mu_n x - \beta\cos\mu_n(1-x)].$$

Lemma 2.3. Spectrum of the operator

$$Bz = -z''(x), \qquad x \in (0,1);$$

$$\begin{cases} \bar{\beta}z(0) + \bar{\alpha}z(1) = 0, \\ \alpha z'(0) + \beta z'(1) = 0, \end{cases}$$

consists of eigenvalues:

$$\mu_n = arccos\left[-\frac{\alpha\bar{\beta} + \bar{\alpha}\beta}{|\alpha|^2 + |\beta|^2}\right] + 2n\pi, n = 0, \pm 1, \pm 2, ...$$

which correspond to the eigenfunctions:

$$z_n(x) = K_n[\alpha\cos\mu_n x - \beta\cos\mu_n(1-x)],$$

where K_n are arbitrary constants.

We note that there is the following equality

$$\lambda_n = \mu_n, \qquad n = 0, \pm 1, \pm 2, ...$$

i.e. spectrums of the operators A and B coincide, and this happens when operators A and B are similar to each other, i.e. there is the equality

$$TA = BT,$$

where $T,\ T^{-1}$ are linear bounded operators.

We find the similarity operator T. We note that

$$l^{+}l = A, \qquad ll^{+} = B$$

thus

$$ll^{+}l = lA, \qquad ll^{+}l = Bl,$$

i.e. $lA = Bl$, but the operator l is unbounded, therefore it is not suitable for our purposes.

As a suggestive idea, we take the following Putnam theorem.

Theorem [1. p.337]. Let $M, N, T \in \mathcal{B}(H)$, moreover the operators M, N are normal, and the operator T is invertible. We suppose that

$$M = TNT^{-1}. \tag{1.9}$$

If $T = UP$ is a polar decomposition of the operator T, then

$$M = UNU^{-1}.$$

Two operators, connected by the relation (1.9), are called similar. If U is a unitary operator and the relation (1.9) holds, then the operators M and N are called unitarily equivalent. Thus, in this theorem we establish that similar normal operators are unitarily equivalent.

Our operators A, B are Hermitian (i.e., symmetric), and such operators belong to the class of normal operators; therefore, there exists a unitary operator T such that

$$AT = TB.$$

We assume that exactly this operator is a solution of the equations:

$$(Tl)^2 = l^{+}l = A, \qquad (lT)^2 = ll^{+} = B.$$

Perhaps, it is necessary to impose additional conditions to the operator T?!

From the formula

$$(Tl)^2 = Tl \cdot Tl = l^{*}l,$$

we see that it is necessary to require $Tl = l^{*}T^{*}$, then

$$Tl \cdot Tl = l^{*}\underbrace{T^{*}T}_{I}l = l^{*}l = A,$$

Further, from $Tl = l^{*}T^{*}$, we get

$$TlT = l^{*}, \qquad lT = T^{-1}l^{*} = T^{*}l^{*},$$

then

$$(lT)^2 = lT \cdot lT = |lT = T^{*}l^{*}| = lT \cdot T^{*}l^{*} = ll^{*} = B.$$

Moreover,

$$TB = Tll^* = l^*T^*l^* = l^*lT = AT.$$

We have proved the following lemma.

Lemma 2.4. If T is an unitary operator satisfying the condition

$$(Tl)^* = Tl = l^*T^*,$$

then the following formulas hold:

a) $(Tl)^2 = l^*l = A$,

b) $(lT)^2 = ll^* = B$,

c) $AT = TB$.

Therefore, the problem has come down to finding a unitary operator T, with the property $Tl = l^*T^*$.

We build an unitary operator T, satisfying the condition:

$$Tl = l^*T^*.$$

If $y \in D(l)$, then $z(x) = T^*y(x) \in D(l^*)$. We look for the operator T as follows

$$T = i\cos\varphi\, I + \sin\varphi \cdot S,$$

where $Su(x) = u(1-x)$, I is unit operator, φ is unknown (yet) angle, then

$$T^* = -i\cos\varphi\, I + \sin\varphi \cdot S,$$

$$TT^* = \cos^2\varphi \cdot I + i\cos\varphi \cdot \sin\varphi \cdot S - i\cos\varphi\sin\varphi \cdot S + \sin^2\varphi \cdot I = I,$$

$$y \in D(l),\ z(x) = T^*y(x) \in D(l^*)$$

$$z(x) = T^*y(x) = -i\cos\varphi\, y(x) + \sin\varphi\, y(1-x),$$

$$\bar{\beta}[-i\cos\varphi\, y(0) + \sin\varphi\, y(1)] + \bar{\alpha}[-i\cos\varphi\, y(1) + \sin\varphi\, y(0)] =$$

$$\left(-i\cos\varphi \cdot \bar{\beta} + \bar{\alpha}\sin\varphi\right)y(0) + \left(\bar{\beta}\sin\varphi - i\bar{\alpha}\cos\varphi\right)y(1) = 0.$$

Let's make the equation scheme:

$$\begin{cases} \left(-i\cos\varphi \cdot \bar{\beta} + \bar{\alpha}\sin\varphi\right)y(0) + \left(\bar{\beta}\sin\varphi - i\bar{\alpha}\cos\varphi\right)y(1) = 0, \\ \alpha y(0) + \beta y(1) = 0. \end{cases}$$

Calculate the determinant of this system of equations

$$\Delta = \begin{vmatrix} -i\cos\varphi \cdot \bar{\beta} + \bar{\alpha}\sin\varphi & \bar{\beta}\sin\varphi - i\bar{\alpha}\cos\varphi \\ \alpha & \beta \end{vmatrix} = 0,$$

$$-i\cos\varphi \cdot |\beta|^2 + \beta\bar{\alpha}\sin\varphi - \alpha\bar{\beta}\sin\varphi + i|\alpha|^2\cos\varphi = 0;$$

$$i(|\alpha|^2 - |\beta|^2)\cos\varphi + \left(\beta\bar{\alpha} - \alpha\bar{\beta}\right)\sin\varphi = 0.$$

a) If $\alpha\bar{\beta} - \bar{\alpha}\beta \neq 0$, then

$$tg\varphi = \frac{i(|\alpha|^2 - |\beta|^2)}{\alpha\bar{\beta} - \bar{\alpha}\beta};$$

b) If $|\alpha|^2 - |\beta|^2$, then

$$ctg\varphi = \frac{\alpha\bar{\beta} - \bar{\alpha}\beta}{i(|\alpha|^2 - |\beta|^2)}.$$

c) If $|\alpha|^2 - |\beta|^2 = 0$ and $\alpha\bar{\beta} - \bar{\alpha}\beta = 0$, then

$$\begin{cases} \alpha\bar{\alpha} - \beta\bar{\beta} = 0, \\ \alpha\bar{\beta} - \bar{\alpha}\beta = 0, \end{cases} => \begin{vmatrix} \bar{\alpha} & -\bar{\beta} \\ \bar{\beta} & -\bar{\alpha} \end{vmatrix} = -(\bar{\alpha})^2 + (\bar{\beta})^2 = 0, => \alpha^2 - \beta^2 = 0,$$

the converse is also true.

In our case the case c) holds, therefore the operator T has the form:

$$T = i\cos\varphi\, I + \sin\varphi \cdot S,$$

where $0 \le \varphi \le 2\pi$ is an arbitrary angle.

3. Research Results.

Theorem 3.1. The following formulas hold:

a) $(Tl)^2 = l^*l = A$,

b) $(lT)^2 = ll^* = B$,

where

$$Ay = -y''(x),\ x \in (0,1);$$

$$\alpha y(0) + \beta y(1) = 0,\ \bar{\beta} y'(0) + \bar{\alpha} y'(1) = 0;$$

$$Bz = -z''(x) = \mu^2 z(x),\ x \in (0,1);$$

$$\bar{\beta} z(0) + \bar{\alpha} z(1) = 0, \alpha z'(0) + \beta z'(1) = 0;$$

$$ly = y'(x),\ x \in (0,1); \tag{1.10}$$

$$\alpha y(0) + \beta y(1) = 0, |\alpha| + |\beta| \neq 0; \tag{1.11}$$

$$l^+ z = -z'(x),\ x \in (0,1), \tag{1.10$^+$}$$

$$\bar{\beta} z(0) + \bar{\alpha} z(1) = 0; \tag{1.11$^+$}$$

$$T = i\cos\varphi\, I + \sin\varphi \cdot S, \tag{1.12}$$

$$Su(x) = u(1 - x);$$

The angle φ is defined by the following way:

If $\alpha\bar{\beta} - \bar{\alpha}\beta \neq 0$, Then

$$tg\varphi = \frac{i(|\alpha|^2 - |\beta|^2)}{\alpha\bar{\beta} - \bar{\alpha}\beta}; \tag{1.13}$$

If $|\alpha|^2 - |\beta|^2 \neq 0$, then

$$ctg\varphi = \frac{\alpha\bar{\beta} - \bar{\alpha}\beta}{i(|\alpha|^2 - |\beta|^2)}; \quad (1.14)$$

If $\alpha^2 - \beta^2 = 0$, then φ is an arbitrary angle, belonging to $0 \leq \varphi \leq 2\pi$.

4. Discussion.

If $\alpha\bar{\beta} - \bar{\alpha}\beta \neq 0$ and $|\alpha|^2 - |\beta|^2 \neq 0$, then from (1.13) we obtain $tg\varphi = 0$, thus $\varphi = 0$, then (1.12) implies that

$$Tly = iy'(x), \ x \in (0,1), \quad (1.14)$$

$$\alpha y(0) + \beta y(1) = 0,$$

where $|\alpha|^2 - |\beta|^2 = 0$.

If $|\alpha|^2 - |\beta|^2 \neq 0$ and $\alpha\bar{\beta} - \bar{\alpha}\beta = 0$, then from the formula (1.14) we have $ctg\varphi = 0$, then $\varphi = \frac{\pi}{2}$, and from the formula (1.12) we obtain that

$$Tly = Ty' = Sy'(x) = y'(1-x), \quad (1.15)$$

$$\alpha y(0) + \beta y(1) = 0, |\alpha| + |\beta| \neq 0, \quad (1.16)$$

where $\alpha\bar{\beta} - \bar{\alpha}\beta = 0$.

This last operator (1.15) - (1.16) has been studied in detail in [67], the results of which were used in study of the Gourse problem for wave equation [68] and in solving inverse problems [69-70]. These two spectral problems are special cases of the spectral problem $Tly = \lambda y$, $\alpha y(0) + \beta y(1) = 0$, $|\alpha| + |\beta| \neq 0$, study of which is of undoubted interest in all respects.

5. Conclusion.

Square root of the Sturm-Liouville operator is a functional differential operator of the first-order that generalizes the well-known momentum operator. Results of this paper can be used to solve inverse problems of mathematical physics, as well as in the spectral theories of linear operators.

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ШТУРМ - ЛИУВИЛЛ ОПЕРАТОРЫНЫҢ КВАДРАТ ТҮБІРІ

Аннотация. Бұл еңбекте Штурм-Лиувилл операторының квадрат түбірі табылды және бұл түбірдің бірінші ретті функционәл-дифференциал оператор екені көрсетілді. Бұл функционәл-дифференциал операторға сәйкес шекаралық есеп табылды. Бағдар ретінде Путнамның бір теоремасы қолданылды. Штутм-Лиувилл операторының шекаралық шарты онша кең емес, оның түр зерттеу әдісіне тәуелді. Табылған унитар оператор көпке әйгілі импуліс операторының ширатылған немесе кеңейтілген түрі десек-те болады.

Түйін сөздер. Штутм-Лиувилл операторы, оператрдың квадрат түбірі, функционәл-дифференциал оператор, аргументі ауытқыған теңдеулер, Катоның гипотезасы, Макинтоштың мысалы, Гурсаның операторы, кері есеп, спектр, меншікті мәндер, меншікті фукнциялар, унитар опратор, ұқсастық операторы.

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О КВАДРАТНОМ КОРНЕ ИЗ ОПЕРАТОРА ШТУРМА - ЛИУВИЛЛЯ

Аннотация. В данной работе найден корень квадратный из оператора Штурма - Лиувилля и показан, что этот корень является функционально- дифференциальным оператором первого порядка. Найден вид соответствующей краевой задачи этого функционально - дифференциального уравнения. В качестве наводящей идеи использована одна теорема Путнама. Краевые условия оператора Штурма - Лиувилля имеют весьма специальный вид, и они продиктованы методом исследования. Найденный унитарный оператор обобщает известного оператора импульса.

Ключевые слова: оператор Штурма - Лиувилля, квадратный корень из оператора, функционально-дифференциальный оператор, уравнения с отклоняющимся аргументом, гипотеза Като, пример Макинтоша, оператор Гурса, обратная задача, спектр, собственные значения, собственные функции, унитарный оператор, оператор подобия.

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